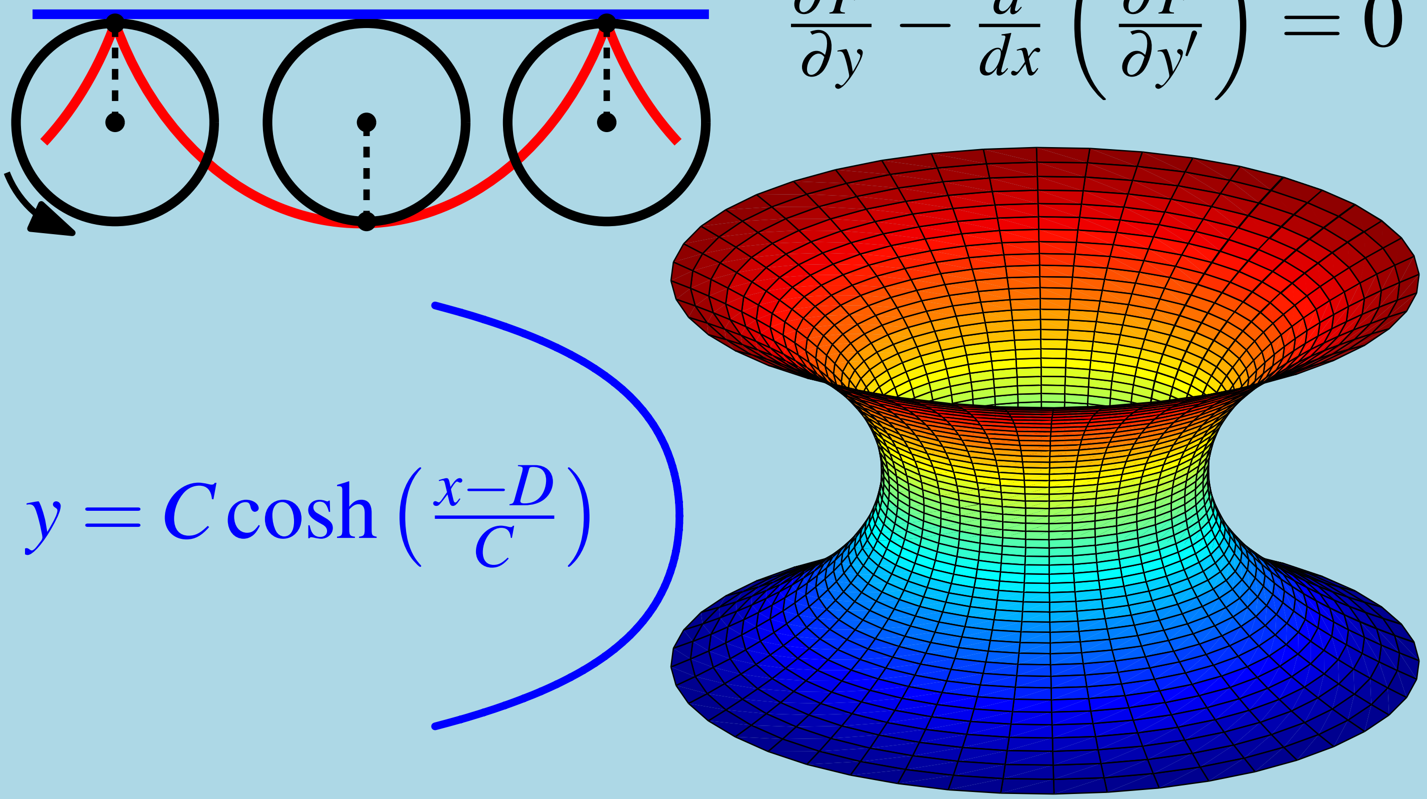
Introduction to the Mathematics of Variation

$$x = a(\phi - \sin\phi)$$
$$y = a(1 - \cos\phi)$$

$$I[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$



$$y = C\cosh\left(\frac{x-D}{C}\right)$$

Taha Sochi

Preface

This book is about the calculus of variations which is a subject concerned mainly with optimization of functionals. However, because part of it is based on using ordinary calculus in solving optimization problems, “Calculus of Variations” in its original title is modified to become “Mathematics of Variation”. In fact, the book is essentially a collection of solved problems with rather modest theoretical background and hence it is based on the method of “learning by example and practice” which in our view is the most effective way for learning mathematics and overcoming the difficulties of its abstraction. The main merit of the book is its clarity, intuitive structure and rather inclusiveness as it includes the main topics and applications of this subject. The structure of the book is that it starts with a preliminary chapter which provides a general theoretical background about this subject with many solved problems related to this background. In the remaining chapters we present a number of common topics and applications related to the mathematical methods of variation. So, the remaining chapters consist essentially of solved problems classified according to certain mathematical and physical criteria with some introductory and general background.

Because the book is about the mathematical methods of variation, we do not explain things like integration or partial differentiation or how to obtain solutions of differential equations although we usually make short explanatory remarks or put the results in a format that is easy to understand and verify. We also avoid going through many theoretical details and technicalities of the calculus of variations to avoid unnecessary distraction from our main practical objectives and to save space. For example, we do not go through the derivation of the formulae of the calculus of variations because such details can be found in almost every book on this subject. Similarly, we avoid going through technicalities related for example to the nature of the arguments of real-valued functions (such as the square root or natural logarithm) and how and when these arguments should be non-negative. So, in brief we take things rather easy in presenting the subject using general understanding and common sense in dealing with the variational problems. Accordingly, the materials in this book have essentially practical objectives rather than pedantic mathematical purposes. So, the target of this book is essentially scientists, engineers and applied mathematicians rather than mathematicians.

The materials in this book require decent background in general mathematics (mostly in single-variable and multi-variable differential and integral calculus). Some problems and applications also require reasonable background in physics which is the main field of application of the calculus (and mathematics) of variations. The book can be used as a text or as a reference for an introductory course on this subject as part of an undergraduate curriculum in physics or engineering or applied mathematics. The book can also be used as a source of supplementary pedagogical materials used in tutorial sessions associated with such a course.

Taha Sochi
London, October 2020

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Nomenclature

In the following list, we define the common symbols, notations and abbreviations which are used in the book as a quick reference for the reader. The list may exclude what is used locally and casually.

∇	nabla differential operator
∇f	gradient of scalar f
$'$ (prime)	total derivative (usually with respect to coordinate x, i.e. d/dx)
$\dot{}$ (overdot)	total derivative (usually with respect to time t, i.e. d/dt)
$\partial\Omega$	boundary of domain Ω
a, b	lengths of semi-major and semi-minor axes of ellipse
a, b, c	lengths of semi-axes of ellipsoid
c	the speed of light in vacuum
c_i	parameters in 1D Rayleigh-Ritz method
c_{ij}	parameters in 2D Rayleigh-Ritz method
c_m	the speed of light in material media
C, D, E	constants
$C_1, \cdots, C_i$	constants
ds	infinitesimal line element of curve
e	eccentricity of ellipse
Eq., Eqs.	Equation, Equations
f	function
F	integrand of functional integral
g	magnitude of gravitational acceleration or gravitational force field
g, $g_1, \cdots, g_i$	constraints (in Lagrange multipliers technique)
G, $G_1, \cdots, G_i$	integrands of constraints (in Lagrange multipliers technique)
h	function to be optimized with constraint (in Lagrange multipliers technique)
h, H	height
H	integrand of constrained variational functional (in Lagrange multipliers technique)
I	moment of inertia
$I[y]$	functional integral (optimized or stationarized by y)
l	length
L	Lagrangian (in Hamiltonian mechanics)
L, W	length, width
m, M	mass
m_r	reduced mass
n	refractive index of light in material media
p	perimeter
r, R	radius
$\mathbf{r}$	position vector
$\mathbf{R}$	position vector of the center of mass
$\mathcal{R}$	resistance to fluid flow
$\mathbb{R}$	the set of real numbers
r, θ, ϕ	spherical coordinates of 3D Euclidean space
s	length of curve
S	surface
$S\,[y_1, \cdots, y_n]$	sum (representing discrete form of functional integral) stationarized by $y_1, \cdots, y_n$
t	time
T	kinetic energy
T_r, T_t	rotational energy, translational energy
U	potential energy

v	speed
v_w	speed of wave
$\mathbf{v}$	velocity
V	volume
x, y, z	coordinates of 3D Euclidean space (usually orthonormal Cartesian)
y_i	the i^{th} 1D Rayleigh-Ritz approximation
$y_1, \cdots, y_n$	y values of function points in 1D finite difference method[1]
$y^{(i)}$	the i^{th} derivative of y (i.e. $y^{(i)} = d^i y/dx^i$)
y_{ji}	partial derivative of y_j with respect to x_i (i.e. $y_{ji} = \partial y_j/\partial x_i$)
y_{x_1}	partial derivative of y with respect to x_1 (i.e. $y_{x_1} = \partial y/\partial x_1$)
z_{mn}	the mn^{th} 2D Rayleigh-Ritz approximation
$z_{11}, \cdots, z_{mn}$	z values of function points in 2D finite difference method
Γ	curve
λ, $\lambda_1, \cdots, \lambda_n$	Lagrange multipliers, eigenvalues
λ_m, λ_M	minimum, maximum eigenvalue of Sturm-Liouville problem
μ	linear mass density
ρ, ϕ	polar coordinates of plane
ρ, ϕ, z	cylindrical coordinates of 3D Euclidean space
σ	area
τ	tension force
$\phi_0, \cdots, \phi_n$	basis functions in 1D Rayleigh-Ritz method[2]
Φ	gravitational potential
ω	angular speed
Ω	set of functions, domain of multi-variate function

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[1] In fact, these y values (as well as the upcoming z values $z_{11}, \cdots, z_{mn}$) are supposed to be approximates of the extremizing function at these points.

[2] In fact, we should also have $\phi_0, \phi_{11} \cdots, \phi_{mn}$ basis functions for the 2D Rayleigh-Ritz method but we did not use this notation in this book.

Chapter 1
Preliminaries

In this chapter we provide an outline of the calculus of variations and its basic principles. In the subsequent chapters we will investigate various categories of variational problems using different methods and approaches which are mostly based on what is outlined in this chapter. However, before we start we make a few introductory remarks of general nature.

1.1 Introductory Remarks

In this section we present a number of general remarks related to conventions, notations and commonly occurring issues in this book.

• As indicated in the Preface, we have no primary interest in the theory of the calculus of variations (or the mathematics of variations to be more general) due to the fact that this book is mainly of practical nature related to solving variational problems rather than presenting and justifying theories and deliberating on abstract issues. However, we provided summaries of the required theoretical background if and when needed.

• Because of the nature and objective of the book (as outlined in the previous remark) we do not go during our investigations and problem solving through details and technicalities not related directly and strongly to the calculus of variations such as the restrictions on the domain of validity of functions or the constraints on the obtained solutions and roots or the methods of solving differential equations.

• Although the book is supposed to be about the calculus of variations, in many problems we use ordinary calculus to solve the variational problems and hence the subject of the book is the mathematics of variations with the calculus of variations being the main subject.

• The terms of 2D shapes (such as “circle”, “rectangle” and “square”) may be used conveniently as surfaces or as curves (defined by their perimeters). Similarly, the terms of 3D shapes (such as “sphere”, “cube” and “parallelepiped”) may be used conveniently as solids and as surfaces. A matter related to this issue is the use of “surface of revolution” and “solid of revolution” (where the latter is usually used when volume is involved).

• Some terms may be used conveniently to represent an object or its quantitative measure. For example, “perimeter” may be used to mean the curve or to mean its length and hence we may say “this point is on the perimeter” or “the perimeter is equal to 10”. This is to ease the text and avoid unnecessary complications and distractions. Accordingly, the context should always be taken into account to determine the ultimate meaning.

• All the functions and functionals in this book are assumed to be sufficiently smooth and differentiable (at least piecewise).

• All the mathematical quantities in this book are real (i.e. not imaginary or complex). Accordingly, all the arguments of real-valued functions that are not defined for negative quantities, like square roots and logarithmic functions, are assumed non-negative by taking the absolute value, if necessary, without using the absolute value symbol. We also exclude any potential mathematical absurdity like division by zero.

• In solving the problems in this book we deliberately used different methods and approaches when we have the freedom to do so. The purpose of this is to diversify and provide the reader with more thorough techniques in tackling the variational problems. In fact, in some cases we solved some problems more than once using different methods of solution (such as using different variational approaches or different coordinate systems).

• Apart from the basic and fundamental notation and symbolism of the calculus of variations (including calculus), we did not commit ourselves to adopting unified notation and symbolism across the entire book for secondary objects and concepts. In fact, in some cases we deliberately used different notations in different Problems for these secondary items for the purpose of distinction and to avoid potential confusion. We also deliberately diversified some secondary symbols and notations in some cases for the purpose of training the reader to deal with different symbols and notations so that he acquires the ability and skill to capture and recognize the essential mathematical forms and patterns regardless of the specific symbols and notations used to express and formulate these forms and patterns in the individual mathematical problems. Anyway, the reader can always consult the Nomenclature for providing consistent (although potentially generic, tentative and non-comprehensive) explanation of the used symbols if no such explanation is provided locally.

- The Problems in the chapters and sections represent just a small sample of the categories represented by these chapters and sections. We did our best to make the selected sample representative of the entire category so that it reflects the main aspects and the most important features of that category (within the restrictions on the book size and scope).
- For optimal categorization of the Problems and to ensure better structures, we may refer in some cases to Problems or items (e.g. graphs) in later parts of the book. However, we ensured that there is no ambiguity or difficulty in understanding and appreciating the indicated content.
- In this book we are mainly interested in optimization problems and techniques regardless of the nature of the optimum and if it is maximum or minimum (and whether it is local or global).[3] Therefore, we generally do not investigate the nature of the optimal technically (e.g. by investigating the signs of the derivatives) although we usually provide short remarks based on intuitive and simple (but not rigorous or technical) arguments about the nature of the obtained optimal. In fact, we even ignored the investigation of the possibility of having non-optimal stationary solutions (e.g. inflection or saddle points) in some cases.[4]
- The axes of most plots in this book are not scaled equally. Also, the purpose of the figures in this book is to demonstrate and outline the main features and settings of interest in the concerned problems, and hence these figures may not be realistic in their shapes and dimensions.

1.2 The Calculus of Variations and the Variational Principle

The calculus of variations is a branch of mathematics whose purpose is to find methods and techniques for optimizing (i.e. minimizing and maximizing) functionals (and possibly functions as well as special cases). The methods and techniques are commonly analytical where the tools and techniques of single-variable and multi-variable differential and integral calculus are used extensively. However, these methods and techniques have usually numerical implementations and instantiations as well (see for example § 9). This branch of mathematics is exceptionally beautiful and motivating. One reason for this is that most problems of this subject have physical roots as they originate from real physical issues and challenges. In fact, the calculus of variations is exceptionally important to physicists because variational principles and paradigms permeate the entire physics and they are at the heart and foundation of many physical problems and issues. This has also historical roots because most of the ideas and techniques of this subject were developed and cultivated within physical contexts and environments to meet urgent demands for solving real-life physical problems. Therefore, physicists and engineers in particular must acquire sufficient knowledge and familiarity with this vital and charming subject.

The entire structure of the calculus of variations (and indeed any variational technique) is based on a very simple principle which we call the variational principle. The essence of this principle is that the variation of any function at its extremums vanishes.[5] So, to find the extremums (or optimums) of a given function where the function reaches its maximum or minimum values (usually locally) we should search for the stationary values of the function where its derivative(s) vanish. This principle is intuitive because at an extremum the function should change its trend (i.e. from increasing to decreasing or from decreasing to increasing) and hence at the extremum the function should cease to vary abruptly. In other words, a positive/negative variation trend followed by a negative/positive variation trend should be separated by zero variation.

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[3] This is mainly to reduce the size of the book and provide more space for the main objectives of the mathematics of variation (noting that the technical investigation of the nature of optimums is usually lengthy and technically messy).

[4] In this regard, the reader should note that in some variational problems optimal solution may not exist (within the given conditions and circumstances).

[5] In this context, "function" is more general than "simple function" and "function of function" where the latter is known as functional (see § 1.3). We should also note that although "extremum" suggests maximum or minimum, it is used in some texts of variational calculus to include even inflection (or stationary) values.

As we will see, optimization problems generally require the methods and techniques of the calculus of variations for their solution. However, some simple optimization problems can be solved by ordinary calculus with no need for the calculus of variations.[6] But even these simple problems usually require a variational argument (which is based ultimately on the variational principle) for their solution (see for example the Problems of § 2.3 and § 5). In fact, some of these simple problems can also be solved using the formal techniques of the calculus of variations although this usually incurs an extra cost and hence it should be avoided (unless it is needed for legitimate purposes).

Problems

1. What is the main objective of the calculus of variations?
   Answer: The main objective is to find and develop mathematical methods and techniques for optimization of functionals (i.e. finding their maximums and minimums).
2. In the text we described the calculus of variations as a branch of mathematics whose purpose is to find methods and techniques for *optimizing* functionals. Comment on *optimizing*.
   Answer: To be more general and inclusive we should replace "optimizing" (or extremizing) with "finding stationary values" (or what we may call "stationarizing"). However, optimization is the main objective in the investigations of this subject and hence almost all the problems of the calculus of variations are related to optimization. In fact, most problems in the calculus of variations are related to minimization.
   Note: in this book we commonly use terms like extremizing or optimizing to mean stationarizing and we rely on this understanding (noting that extremizing and optimizing are more common in use and these cases are more frequent in occurrence in the real life and hence they are the main focus of investigation).

1.3 Functionals in the Calculus of Variations

In simple terms, a functional is a function of function, e.g. $\phi\left[f(x)\right]$ where ϕ is a functional that depends on f which is a function of x. So, a functional $\phi\left[f(x)\right]$ in an interval $[x_1, x_2]$ is a function that depends on all the values of another function $f(x)$ in that interval. A functional in an interval may also be described as a map from functions defined on this interval to the real numbers. In the calculus of variations (of one variable), the functionals that are commonly dealt with have the following form:

$$I\left[y\right] = \int_{x_1}^{x_2} F\left(x, y, y'\right)\, dx \tag{1}$$

where I is a functional of the function $y \equiv y\left(x\right)$, $y' = dy/dx$, $y\left(x_1\right) = C_1$ and $y\left(x_2\right) = C_2$ (with C_1 and C_2 being given numbers). The notation $[y]$ indicates that the variation and extremization of I is essentially determined by the form of y while the notation (x, y, y') indicates the dependencies of F (which is the integrand). We should also note that the choice of I to symbolize the functional is to indicate the fact that the functional is an integral (and hence we usually call I in this book the functional integral). To summarize the essence of Eq. 1 we can say: the functional I whose variation/optimization depends on the nature of y (which is a function of x) is the integral of F (which is a function of x, y, y') over the real interval $[x_1, x_2]$.

It should be noted[7] that in the formulations and applications of the Euler-Lagrange equation (which is the essence of the calculus of variations and is based on the functional of Eq. 1 as will be investigated next) the symbols x, y, y' are treated as if they are representing variables that are independent of each other

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[6] In fact, some may be solved even by simple arithmetic and algebraic techniques based on simple intuitive arguments.

[7] In fact, this note applies to the form of Eq. 1 (i.e. single independent and dependent variables). For other forms (e.g. the form of Eq. 14) more details are required (as will be seen in the upcoming chapters, sections, Problems and applications).

(although in reality y is usually dependent on x and y' is usually dependent on x and possibly y) and this is reflected in taking partial derivatives with respect to these variables (as will be seen in the upcoming examples and applications). It is also important to note that the symbols x, y, y' should be seen as generic symbols and hence they are not necessarily representing variables in Cartesian systems. For example, these symbols can represent ϕ, ρ, ρ' in 2D polar coordinates or ϕ, z, z' in 3D cylindrical coordinates (where the prime represents $d/d\phi$) or $t, x, \dot{x}$ in mechanics (where t is time, x is spatial coordinate and $\dot{x} \equiv dx/dt$).

Problems

1. Give a brief symbolic definition for "functional" as a mapping.
   Answer: A functional ϕ is a function defined by the mapping $\phi : \Omega \to \mathbb{R}$ where Ω is a set of functions and $\mathbb{R}$ is the set of real numbers.
2. Make a simple comparison between functions and functionals as mapping devices.
   Answer: Functions map numbers to numbers while functionals map functions to numbers.
3. Make a simple comparison between optimization of functions and optimization of functionals considering the number of variables involved in the optimization process.
   Answer: The first is an optimization of functions of finite number of variables while the second can be seen as an optimization of *functions* of infinite number of variables.
4. Compare the functional notation $I[y]$ to the function notation $f(x)$ and outline the significance of each notation. Also, outline the difference between functional variation (in functional relations) and function variation (in function relations) at stationary values.
   Answer: The use of square brackets in $I[y]$ is to distinguish functional dependency (i.e. the *functional* I depends on the function y) from function dependency in $f(x)$, i.e. the *function* f depends on the independent variable x. So, in $I[y]$ the value of I depends on the form of y on the entire interval $[x_1, x_2]$ while in $f(x)$ the value of f at a certain point depends on the value of x at that point.
   In functional relations the variation of y anywhere in the interval $[x_1, x_2]$ will disturb the stationary value of the functional (and hence the stationary value of the functional is obtained from a single form of y over the entire interval), while in function relations the variation of x away from the immediate neighborhood of the stationary point does not disturb the stationary value of the function (i.e. the function and its stationary point have only local dependency).
5. Briefly explain the objective in the problems investigated by the calculus of variations and how they are dealt with.
   Answer: The objective is to find the form of a function $y = f(x)$ such that a definite integral I of a given expression F that involves y and its derivative $y' \equiv dy/dx$ is extremum.
   To deal with these problems we start by forming the integral:
   $$I\,[y] = \int_{x_1}^{x_2} F\,(x, y, y')\;dx$$
   where F is formulated according to the description and statement of the problem. We then use the techniques of the calculus of variations to find the solution y. Any unknown parameters in the solution may then be determined by using the given boundary conditions and constraints. The main step in the solution technique of the calculus of variations is the formation of the Euler-Lagrange equation from the obtained F where this equation (in its various shapes and forms depending on the nature of the problem and the form of F) is solved subject to the given boundary conditions and constraints to obtain the final solution.
6. What is the difference between the variational problems in ordinary calculus and the variational problems in the calculus of variations.
   Answer: In ordinary calculus the variational problems are about finding points (x_i, y_i) that extremize (or stationarize) a given function $y = f(x)$, while in the calculus of variations the variational problems are about finding functions $y(x)$ (represented by curves) that extremize (or stationarize) a given functional $I\,[y]$. So, in ordinary calculus we are looking for points while in the calculus of variations we are looking for curves (represented by functions).[8] This is demonstrated schematically in Figure 1.

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[8] In other words, in ordinary calculus (the form of) y is known (e.g. $y = x^2 + 1$) but its extremum (or stationary) points are

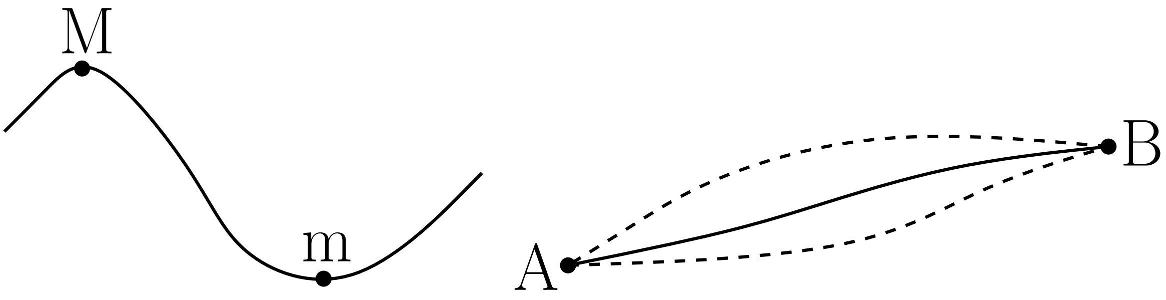

Figure 1: A schematic illustration of the difference in the nature of the variational problems in ordinary calculus and in the calculus of variations where on the left (representing ordinary calculus) the objective is to find the extremal points (M and m) on the curve, while on the right (representing the calculus of variations) the objective is to find the extremal curve (solid) among all other curves (dashed) in its immediate neighborhood that connect two given points (A and B). See Problem 6 of § 1.3.

1.4 The Euler-Lagrange Equation

The Euler-Lagrange equation is a relation that implements the variational principle in a specific mathematical form when the function that should be optimized is a functional. More specifically, the Euler-Lagrange equation is a mathematical relation whose objective is to minimize or maximize[9] a certain functional $I[y]$ which depends on a function y by finding this y. It is represented mathematically by a differential equation whose solution(s) optimize the particular functional I. The Euler-Lagrange equation in its generic, simple and most common form is given by:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{2}$$

where F (which is a function of x, y, y') is the integrand of a functional integral $I[y]$ whose variation/optimization depends on y (which is a function of x), and y' is the derivative of y with respect to x (i.e. $y' = dy/dx$). It can be shown (see Problem 3) that the functional given by Eq. 1 (see § 1.3) possesses stationary values obtained by solving the above Euler-Lagrange equation (i.e. Eq. 2) where “solving” means obtaining the form of $y(x)$. So in brief, the objective of the variational problems in the calculus of variations (as formulated by the Euler-Lagrange equation) is to find the function y that extremizes (i.e. minimizes or maximizes) the functional I.[10]

The Euler-Lagrange equation is simplified in the following cases:

I. If F does not depend explicitly on x $\left[\text{i.e. } F \equiv F(y, y')\right]$ the Euler-Lagrange equation (Eq. 2) will reduce to the following form:[11]

$$F - y'\frac{\partial F}{\partial y'} = C \tag{3}$$

where C is a constant. This equation is called the Beltrami identity (see Problem 5).

II. If F does not depend explicitly on y $\left[\text{i.e. } F \equiv F(x, y')\right]$ then $\frac{\partial F}{\partial y} = 0$ and hence Eq. 2 takes the

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unknown and hence we are required to find these points, while in the calculus of variations (the form of) y that extremizes (or stationarizes) I is unknown (e.g. whether $y = x^2 + 1$ or $y = e^x$ or $y = \sin x$, etc.) and hence we are required to find (the form of) y that extremizes I (so that a function of any other form or a function that is obtained by a slight variation of y in its neighborhood will make I deviate from its extremum value).

[9] In fact, a more general objective of the equation is to search for stationary points (although they are usually maximums or minimums).

[10] To be more general, we may need to replace “extremize” with “stationarize” (i.e. find stationary values). However, we do not go through these details.

[11] This equation (and its alike of upcoming equations) is commonly known in the literature of the calculus of variations as a *first integral* of the Euler-Lagrange equation because only one integration is needed to obtain the solution.

following form:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \qquad \text{and hence} \qquad \frac{\partial F}{\partial y'}=C \tag{4}$$

where C is a constant. This equation also applies when F depends explicitly on y' only.[12]

III. If F does not depend explicitly on y' $\left[\text{i.e. } F\equiv F\left(x,y\right)\right]$ then $\frac{\partial F}{\partial y'}=0$ and hence Eq. 2 reduces to the following form:

$$\frac{\partial F}{\partial y}=0 \tag{5}$$

This equation also applies when F depends explicitly on y only.[13]

It is important to note that using these simplified forms is not mandatory although they usually (but not necessarily) make the solution easier. Accordingly, in the future we will use these simplified forms if they are convenient. We should also note that the full form and the simplified forms usually lead to different forms of equations even after simplification although in principle this should not affect the final solution.

We should finally draw the attention to the following remarks:

1. The Euler-Lagrange equation may also be given in the following forms:

$$\frac{\partial F}{\partial x}-\frac{d}{dx}\left(F-y'\frac{\partial F}{\partial y'}\right) = 0 \tag{6}$$

$$\frac{\partial F}{\partial y}-\frac{\partial^2 F}{\partial x\partial y'}-y'\frac{\partial^2 F}{\partial y\partial y'}-y''\frac{\partial^2 F}{\partial y'^2} = 0 \tag{7}$$

These forms will be investigated in Problems 6 and 7.

2. The Euler-Lagrange equation is a necessary, but not sufficient, condition for extremizing the functional I and hence further investigation to determine the nature of the solution is generally required. However, in most practical variational problems (especially in physics) the required extremizing solution is usually found by just solving the Euler-Lagrange equation with no need for further investigation. In fact, in many cases the nature of the solution (as being minimizing or maximizing or even being inflection) can be easily inferred from the nature of the problem using non-formal arguments and simple intuitive considerations (such as geometric or physical considerations).

   Regarding the practical side of this issue, formal identification (by using technically rigorous tools and methods) of the nature of the obtained solution is generally difficult and requires obtaining and testing derivatives of various orders. Therefore, in this book we generally avoid going through these messy details contenting ourselves with just obtaining the solution (and possibly relying on the context and other intuitive considerations to identify the nature of the solution).

3. A solution of the Euler-Lagrange equation may be called an extremal (or extremal function or extremal curve) which is inline with the nature of most solutions of the variational problems in the calculus of variations (although the previous remark should be taken into consideration).

4. If the integrand F is a total derivative (with respect to x) of a given function of x and y, say $\phi(x,y)$, then the Euler-Lagrange equation is satisfied identically, i.e. any function (of any form) that satisfies the given boundary conditions will satisfy the Euler-Lagrange equation (see part m of Problem 9). The reason is that in this case the value of the functional integral is solely determined by the value of ϕ at the boundaries (and hence any function ϕ that satisfies the given boundary conditions should "optimize" the functional integral regardless of the form of ϕ), that is:

$$I\left[y\right]=\int_{x_1}^{x_2}F\left(x,y,y'\right)\,dx=\int_{x_1}^{x_2}\frac{d\phi}{dx}\,dx=\left[\phi\right]_{x_1}^{x_2}=\phi\big|_{x_2}-\phi\big|_{x_1}$$

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[12] In fact, in this case $\frac{\partial F}{\partial y'}$ is independent of x and y and this should lead to a simpler equation (i.e. $y'=$ constant) that can be used instead of Eq. 4.

[13] In fact, in this case $\frac{\partial F}{\partial y}$ is independent of x and y' and this should lead to a simpler equation (i.e. $y=$ constant) that can be used instead of Eq. 5.

An implication of this is that adding a term (or terms) that is a total derivative of some function to the integrand F will not change the Euler-Lagrange equation and this could lead to simplification in some cases where the additional term(s) can be ignored in the derivation of the Euler-Lagrange equation of the given variational problem (see for example part h of Problem 10 or part j of Problem 11).

5. As indicated earlier (see § 1.3), the symbols x, y, y' in the Euler-Lagrange equation are treated as if they are representing variables that are independent of each other. This means that the partial derivatives in the Euler-Lagrange equation operate on the explicit (but not implicit) dependencies of these variables on each other. For example, $\partial y/\partial x = 0$ according to this type of partial differentiation even though y is a function of x. However, there are some exceptions to this rule where in some cases partial derivatives operate on both the explicit and implicit dependencies (see § 1.6). In brief, when we deal with the Euler-Lagrange equation (in its different forms and flavors) we have two types of partial differentiation where in one type (which is the common type) only explicit dependencies are considered while in another type (which is the exceptional type) both the explicit and implicit dependencies are considered. So, we should be careful about the interpretation and treatment of partial derivatives in the applications of variational calculus.
6. As indicated earlier (see § 1.3), the symbols x, y, y' in the Euler-Lagrange equation should be seen as generic symbols and hence they are not necessarily representing variables in Cartesian systems.
7. The term "Euler-Lagrange equation" may be used to refer to the Euler-Lagrange equation in its general form (i.e. as given by Eq. 2 and its equivalent and reduced forms as well as its upcoming generalized and extended forms) and may be used to refer to the equation obtained from applying the Euler-Lagrange equation in its general form to a particular problem and hence the Euler-Lagrange equation in this case is an instantiation of the Euler-Lagrange equation in its general form (i.e. it is the Euler-Lagrange equation for that particular problem). The meaning should be obvious from the context.

Problems

1. Describe in plain terms a simple typical variational problem and how it is formulated and solved.

Answer: Suppose we have a functional I which usually depends on x and $y(x)$ as well as the derivative $y' = dy/dx$, that is:

$$I = \phi\left(x, y, y'\right) \tag{8}$$

This functional is in the form of an integral and its optimization (or stationarization to be more general) depends on the form of the function y assuming that y is fixed at its two end points A(x_1, y_1) and B(x_2, y_2), i.e. $y(x_1) = y_1 = C_1$ and $y(x_2) = y_2 = C_2$. Moreover, we are interested in optimizing this functional by finding the form of y that achieves this optimization. Accordingly, we modify the notation of Eq. 8 (to make it more explicit and suggestive) to the following:

$$I\left[y\right] = \int_{x_1}^{x_2} F\left(x, y, y'\right) \, dx \tag{9}$$

where the notation $I[y]$ is to suggest that the required optimization of I depends on the form of y (which we are looking for) and where F is the integrand (which depends on x, y, y'). In brief, Eq. 9 suggests that if we use a certain function y (which we are looking for) in the integrand F then the integral I will be optimal. So, our objective now is to find this special (or optimizing) function $y(x)$. To find this optimizing y we form the following Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

using F that we formulated in Eq. 9. We then solve this equation (which in general is a second order differential equation) to find the optimizing solution y. Finally, the solution of a second order differential equation contains two unknown constants (as a result of two integrations) and hence we need the given boundary conditions (i.e. the fixed values of y at the two end points) to find the specific solution to our variational problem.

Note: as indicated in the question, the above description belongs to a simple typical variational problem and hence it can be subject to extensions, generalizations and restrictions in different and more complex situations (as will be investigated later on).

2. State some general properties of the Euler-Lagrange equation.
   Answer: For example:
   - This equation in its various shapes and forms (as investigated earlier and will be investigated further later on) is the main pillar of the mathematics of variation (and specifically the calculus of variations).
   - The existence and uniqueness of solution of this equation is not guaranteed in general and hence there may not be a solution, or there is only one solution, or there are multiple (finite or infinite) solutions.
   - The Euler-Lagrange equation is a necessary, but not sufficient, condition for the existence of optimal solution(s) even if solution(s) to this equation do exist. Accordingly, any obtained solution to this equation should be inspected and assessed to determine its nature and if it satisfies the requirements and meets the objectives (e.g. searching for a minimum). In this regard, mathematical and non-mathematical (e.g. physical) considerations should be taken into account.
3. Derive the Euler-Lagrange equation using the variational principle.
   Answer: In the following we outline a rather simple derivation method of the Euler-Lagrange equation. We start from the functional of Eq. 1, that is:

$$I[y] = \int_a^b F(x, y, y')\,dx$$

where the values of y at the two end points are fixed (noting that $x_1 = a$ and $x_2 = b$ are given constants). Now, if y is perturbed slightly to $y + \delta y$ (where δy is a tiny change or variation in y) then I should also be perturbed to $I + \delta I$ where δI is given by:[14]

$$\begin{aligned}
\delta I &= I[y+\delta y] - I[y] \\
&= \int_a^b F(x, y+\delta y, y'+\delta y')\,dx - \int_a^b F(x,y,y')\,dx \\
&= \int_a^b \left[F(x,y,y') + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] dx - \int_a^b F(x,y,y')\,dx \\
&= \int_a^b \left[F(x,y,y') + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' - F(x,y,y')\right] dx \\
&= \int_a^b \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] dx \\
&= \int_a^b \frac{\partial F}{\partial y}\delta y\,dx + \int_a^b \frac{\partial F}{\partial y'}\delta y'\,dx \\
&= \int_a^b \frac{\partial F}{\partial y}\delta y\,dx + \left[\frac{\partial F}{\partial y'}\delta y\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\,dx \\
&= \left[\frac{\partial F}{\partial y'}\delta y\right]_a^b + \int_a^b \left[\frac{\partial F}{\partial y}\delta y - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\right] dx
\end{aligned}$$

where in line 3 we use first order Taylor expansion, and in line 7 we integrate by parts the second term of line 6. Now, since the two end points are fixed (and hence $\delta y = 0$) the first term in the last line is zero and hence:

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y}\delta y - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\right] dx = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,dx$$

---

[14] The use of δ here to symbolize variation is common (and rather conventional although not compulsory) and should indicate the difference between functional variation and function variation where in the latter d or ∂ are usually used to symbolize variation (see Problem 4 of § 1.3).

By the variational principle, I should be stationary at its extremum and hence $\delta I = 0$ for all possible tiny variations δy in y (i.e. δy is arbitrary). This is true *iff*

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

which is the Euler-Lagrange equation. It should be understood that the last equation applies over the entire interval (except possibly at a few isolated points).

4. Referring to the simplified forms of the Euler-Lagrange equation (as given by Eqs. 3-5), it is common in the literature to say F in these cases is independent of (or does not depend on) x or y or y'. Clarify this issue.
   Answer: This saying means that these variables do not appear explicitly in the expression of F although F is generally dependent on these variables implicitly.
5. Prove the Beltrami identity (Eq. 3).
   Answer: By the chain rule we have:

$$\begin{aligned}\frac{dF}{dx} &= \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx}\\ &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}\frac{dy'}{dx}\\ &= \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}\frac{dy'}{dx}\end{aligned}$$

where the last line is justified by the fact that $\partial F/\partial x = 0$ because F is supposed to be independent of x (i.e. F does not depend explicitly on x). Now, by the Euler-Lagrange equation (i.e. Eq. 2) we have $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ and hence on substituting from this into the last line we get:

$$\begin{aligned}\frac{dF}{dx} &= \left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]y' + \frac{\partial F}{\partial y'}\frac{dy'}{dx}\\ \frac{dF}{dx} &= \frac{d}{dx}\left(\frac{\partial F}{\partial y'}y'\right) && \text{(product rule)}\\ \frac{dF}{dx} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}y'\right) &= 0\\ \frac{d}{dx}\left(F - \frac{\partial F}{\partial y'}y'\right) &= 0\\ F - y'\frac{\partial F}{\partial y'} &= C\end{aligned}$$

Note: the Beltrami identity (Eq. 3) can be obtained more simply from the other form of the Euler-Lagrange equation (i.e. Eq. 6) by setting $\partial F/\partial x = 0$. However, we followed the above method for more practice.

6. Derive Eq. 6 from Eq. 2.
   Answer: We have:

$$\begin{aligned}\frac{dF}{dx} &= \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' && \text{(chain rule)}\\ \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) &= \frac{dy'}{dx}\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) && \text{(product rule)}\end{aligned}$$

Now, if we subtract the first line from the second line we get:

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) - \frac{dF}{dx} = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}y'$$

$$\begin{aligned}
\frac{\partial F}{\partial x} + \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) - \frac{dF}{dx} &= y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}y' \\
\frac{\partial F}{\partial x} - \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) &= -y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \\
\frac{\partial F}{\partial x} - \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) &= 0
\end{aligned}$$

where in the last line we used Eq. 2.
Note: from the above derivation we can see that Eq. 6 is based on Eq. 2 (since we used Eq. 2 in the derivation of Eq. 6). However, strictly speaking Eq. 6 is not another form of the Euler-Lagrange equation (as we described it in the text) although it should be equivalent to it.

7. Show that the Euler-Lagrange equation may be given by the following form (see Eq. 7):

$$F_y - F_{y'x} - y'F_{y'y} - y''F_{y'y'} = 0$$

where the subscripts mean partial derivatives with respect to the variables represented by these subscripts, e.g. $F_{y'x}$ means $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)$.
Answer: We have:

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) &= 0 && \text{(Eq. 2)} \\
\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)\frac{dx}{dx} - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} - \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{dy'}{dx} &= 0 && \text{(chain rule)} \\
\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x\partial y'} - y'\frac{\partial^2 F}{\partial y\partial y'} - y''\frac{\partial^2 F}{\partial y'^2} &= 0 \\
F_y - F_{y'x} - y'F_{y'y} - y''F_{y'y'} &= 0
\end{aligned}$$

8. Use $F(x, y, y') = x^3y + y'^2$ to verify that Eqs. 2, 6 and 7 are equivalent.
Answer: From Eq. 2 we have:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[x^3y + y'^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[x^3y + y'^2\right]\right) &= 0 \\
x^3 - \frac{d}{dx}(2y') &= 0 \\
x^3 - 2y'' &= 0
\end{aligned}$$

From Eq. 6 we have:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[x^3y + y'^2\right] - \frac{d}{dx}\left(\left[x^3y + y'^2\right] - y'\frac{\partial}{\partial y'}\left[x^3y + y'^2\right]\right) &= 0 \\
3x^2y - \frac{d}{dx}\left(x^3y + y'^2 - 2y'^2\right) &= 0 \\
3x^2y - \frac{d}{dx}\left(x^3y - y'^2\right) &= 0 \\
3x^2y - 3x^2y - x^3y' + 2y'y'' &= 0 \\
-x^3y' + 2y'y'' &= 0 \\
x^3 - 2y'' &= 0 && (y' \neq 0)
\end{aligned}$$

From Eq. 7 we have:

$$\frac{\partial}{\partial y}\left[x^3y + y'^2\right] - \frac{\partial^2}{\partial x\partial y'}\left[x^3y + y'^2\right] - y'\frac{\partial^2}{\partial y\partial y'}\left[x^3y + y'^2\right] - y''\frac{\partial^2}{\partial y'^2}\left[x^3y + y'^2\right] = 0$$

$$\begin{aligned} x^3 - \frac{\partial}{\partial x}[2y'] - y'\frac{\partial}{\partial y}[2y'] - y''\frac{\partial}{\partial y'}[2y'] &= 0 \\ x^3 - 0 - 0 - 2y'' &= 0 \\ x^3 - 2y'' &= 0 \end{aligned}$$

So, the three equations are equivalent (at least in this case).
Note: the three equations will lead to the same result even if $y' = 0$ because in this case they will all lead to $x^3 = 0$ (at least as a possibility).

9. Obtain the Euler-Lagrange equations for the following variational integrands:
(**a**) $F(x, y, y') = xy^4$.[15]
(**b**) $F(x, y, y') = xy'^3$.
(**c**) $F(x, y, y') = yy'^2$.
(**d**) $F(x, y, y') = xyy'$.
(**e**) $F(x, y, y') = (e^x y^3)/y'$.
(**f**) $F(t, x, \dot{x}) = x\sqrt{1+\dot{x}}$.[16]
(**g**) $F(t, \theta, \dot{\theta}) = t\sin\theta$.
(**h**) $F(x, y, y') = y^2 e^y$.
(**i**) $F(x, y, y') = \sqrt{1+y'^2}$.
(**j**) $F(x, y, y') = y'^2 - y^2$.
(**k**) $F(x, y, y') = (y^2 - 1)/y'^2$.
(**l**) $F(x, y, y') = y'^2/\sqrt{y^2+y'^2}$.
(**m**) $F(x, y, y') = x^2y' + 2xy$.
Answer:
(**a**) y' does not appear in F and hence we can use Eq. 5, that is:

$$\begin{aligned} \frac{\partial F}{\partial y} &= 0 \\ \frac{\partial}{\partial y}\left(xy^4\right) &= 0 \\ 4xy^3 &= 0 \\ xy^3 &= 0 \end{aligned}$$

(**b**) y does not appear in F and hence we can use Eq. 4, that is:

$$\begin{aligned} \frac{\partial F}{\partial y'} &= C \\ \frac{\partial}{\partial y'}\left(xy'^3\right) &= C \\ 3xy'^2 &= C \\ xy'^2 &= D \qquad (D = C/3) \end{aligned}$$

(**c**) x does not appear in F and hence we can use Eq. 3, that is:

$$\begin{aligned} F - y'\frac{\partial F}{\partial y'} &= C \\ yy'^2 - y'\frac{\partial}{\partial y'}\left(yy'^2\right) &= C \end{aligned}$$

---

[15] Despite the absence of explicit dependency on y' in this expression, we use $F(x, y, y')$ to highlight the overall dependencies (whether explicit or implicit). This is also followed in similar examples and similar missing dependencies (unless stated otherwise to highlight the explicit dependencies only).

[16] We note that the overdot (in this expression and similar expressions) means derivative with respect to t (i.e. d/dt).

$$
\begin{aligned}
yy'^2 - y'\left(2yy'\right) &= C \\
yy'^2 - 2yy'^2 &= C \\
yy'^2 &= D \qquad\qquad (D = -C)
\end{aligned}
$$

(**d**) We use Eq. 2, that is:

$$
\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) &= 0 \\
\frac{\partial}{\partial y}\left[xyy'\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[xyy'\right]\right) &= 0 \\
xy' - \frac{d}{dx}\left(xy\right) &= 0 \\
xy' - y - xy' &= 0 \\
y &= 0
\end{aligned}
$$

(**e**) We use Eq. 2, that is:

$$
\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) &= 0 \\
\frac{\partial}{\partial y}\left[\frac{e^x y^3}{y'}\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[\frac{e^x y^3}{y'}\right]\right) &= 0 \\
\frac{3e^x y^2}{y'} - \frac{d}{dx}\left(-\frac{e^x y^3}{y'^2}\right) &= 0 \\
\frac{3e^x y^2}{y'} + \left(\frac{e^x y^3}{y'^2} + \frac{3e^x y^2 y'}{y'^2} - \frac{2e^x y^3 y''}{y'^3}\right) &= 0 \qquad\qquad \text{(product and chain rules)} \\
\frac{3e^x y^2}{y'} + \frac{e^x y^3}{y'^2} + \frac{3e^x y^2}{y'} - \frac{2e^x y^3 y''}{y'^3} &= 0 \\
\frac{6e^x y^2}{y'} + \frac{e^x y^3}{y'^2} - \frac{2e^x y^3 y''}{y'^3} &= 0 \\
\frac{6y^2}{y'} + \frac{y^3}{y'^2} - \frac{2y^3 y''}{y'^3} &= 0 \qquad\qquad (e^x \neq 0)
\end{aligned}
$$

(**f**) t does not appear in F and hence we can use Eq. 3 (noting that $t, x, \dot{x}$ replaces x, y, y'), that is:

$$
\begin{aligned}
F - \dot{x}\frac{\partial F}{\partial \dot{x}} &= C \\
x\sqrt{1+\dot{x}} - \dot{x}\frac{\partial}{\partial \dot{x}}\left(x\sqrt{1+\dot{x}}\right) &= C \\
x\sqrt{1+\dot{x}} - \dot{x}\left(\frac{x}{2\sqrt{1+\dot{x}}}\right) &= C \\
x\left(\sqrt{1+\dot{x}} - \frac{\dot{x}}{2\sqrt{1+\dot{x}}}\right) &= C \\
\frac{x\left(2+\dot{x}\right)}{2\sqrt{1+\dot{x}}} &= C \\
\frac{x\left(2+\dot{x}\right)}{\sqrt{1+\dot{x}}} &= D \qquad\qquad (D = 2C)
\end{aligned}
$$

(**g**) $\dot{\theta}$ does not appear in F and hence we can use Eq. 5 (noting that $t, \theta, \dot{\theta}$ replaces x, y, y'), that is:

$$
\frac{\partial F}{\partial \theta} = 0
$$

$$\begin{aligned}\frac{\partial}{\partial\theta}(t\sin\theta) &= 0\\ t\cos\theta &= 0\end{aligned}$$

(**h**) Only y appears in F and hence we can use Eq. 5, that is:

$$\begin{aligned}\frac{\partial F}{\partial y} &= 0\\ \frac{\partial}{\partial y}\left(y^2e^y\right) &= 0\\ 2ye^y+y^2e^y &= 0 && \text{(product rule)}\\ y^2+2y &= 0 && (e^y\neq 0)\end{aligned}$$

Hence, $y=0$ or $y=-2$ (which is consistent with footnote [13] since $y=\text{constant}$).
(**i**) Only y' appears in F and hence we can use Eq. 4, that is:

$$\begin{aligned}\frac{\partial F}{\partial y'} &= C\\ \frac{\partial}{\partial y'}\left(\sqrt{1+y'^2}\right) &= C\\ \frac{2y'}{2\sqrt{1+y'^2}} &= C\\ \frac{y'}{\sqrt{1+y'^2}} &= C\\ y'^2 &= C^2+C^2y'^2\\ y' &= \pm\sqrt{\frac{C^2}{1-C^2}}\end{aligned}$$

which is consistent with footnote [12] since $y'=\text{constant}$.
(**j**) x does not appear in F and hence we can use Eq. 3, that is:

$$\begin{aligned}F-y'\frac{\partial F}{\partial y'} &= C\\ \left(y'^2-y^2\right)-y'\frac{\partial}{\partial y'}\left(y'^2-y^2\right) &= C\\ y'^2-y^2-y'\left(2y'\right) &= C\\ -y^2-y'^2 &= C\\ y'^2+y^2 &= D && (D=-C)\end{aligned}$$

(**k**) x does not appear in F and hence we can use Eq. 3, that is:

$$\begin{aligned}F-y'\frac{\partial F}{\partial y'} &= C\\ \left(\frac{y^2-1}{y'^2}\right)-y'\frac{\partial}{\partial y'}\left(\frac{y^2-1}{y'^2}\right) &= C\\ \frac{y^2-1}{y'^2}-y'\left(-2\frac{y^2-1}{y'^3}\right) &= C\\ \frac{y^2-1}{y'^2}+2\left(\frac{y^2-1}{y'^2}\right) &= C\\ \frac{y^2-1}{y'^2} &= D && (D=C/3)\end{aligned}$$

$$Dy'^2 - y^2 + 1 = 0$$

(**l**) x does not appear in F and hence we can use Eq. 3, that is:

$$\begin{aligned}
F - y'\frac{\partial F}{\partial y'} &= C\\
\left(\frac{y'^2}{\sqrt{y^2+y'^2}}\right) - y'\frac{\partial}{\partial y'}\left(\frac{y'^2}{\sqrt{y^2+y'^2}}\right) &= C\\
\frac{y'^2}{\sqrt{y^2+y'^2}} - y'\left(\frac{2y'}{\sqrt{y^2+y'^2}} - \frac{y'^2\left(2y'\right)}{2\left(y^2+y'^2\right)^{3/2}}\right) &= C\\
\frac{y'^2}{\sqrt{y^2+y'^2}} - \frac{2y'^2}{\sqrt{y^2+y'^2}} + \frac{y'^4}{\left(y^2+y'^2\right)^{3/2}} &= C\\
-\frac{y'^2}{\sqrt{y^2+y'^2}} + \frac{y'^4}{\left(y^2+y'^2\right)^{3/2}} &= C\\
-\frac{y'^2\left(y^2+y'^2\right)}{\left(y^2+y'^2\right)^{3/2}} + \frac{y'^4}{\left(y^2+y'^2\right)^{3/2}} &= C\\
-\frac{y^2y'^2}{\left(y^2+y'^2\right)^{3/2}} &= C\\
y^4y'^4 &= C^2\left(y^2+y'^2\right)^3
\end{aligned}$$

(**m**) We use Eq. 2, that is:

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) &= 0\\
\frac{\partial}{\partial y}\left[x^2y' + 2xy\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[x^2y' + 2xy\right]\right) &= 0\\
2x - \frac{d}{dx}\left(x^2\right) &= 0\\
2x - 2x &= 0\\
0 &= 0
\end{aligned}$$

This trivial (but true) result is justified by remark 4 in the text because F is a total derivative of x^2y and hence the Euler-Lagrange equation is satisfied identically. In other words, we do not have a specific function that can be seen as a solution to the Euler-Lagrange equation since any function that satisfies the boundary conditions will satisfy the Euler-Lagrange equation. Hence, the Euler-Lagrange equation is $0 = 0$ and its solution is $y =$ any function (satisfying the boundary conditions).

10. Obtain the Euler-Lagrange equations for the following functional integrals:
    (**a**) $I[y] = \int_{x_1}^{x_2}\left(y'^2 - 4\sqrt{xy}\right)dx$.
    (**b**) $I[x] = \int_{t_1}^{t_2}\left(tx^2 + \dot{x}^2\right)dt$.
    (**c**) $I[\phi] = \int_{t_1}^{t_2}(t\cos\phi + t^2\dot{\phi})\,dt$.
    (**d**) $I[y] = \int_{x_1}^{x_2}\left(y'^2 - xy'^2\right)dx$.
    (**e**) $I[x] = \int_{t_1}^{t_2}\left(t^2\dot{x} - x^2\right)dt$.
    (**f**) $I[y] = \int_{x_1}^{x_2}\left(y'^2 - ye^{ax} + y^2\right)dx$ (where a is a constant).
    (**g**) $I[y] = \int_{x_1}^{x_2} y'^3\,dx$.
    (**h**) $I[y] = \int_{x_1}^{x_2}\left(y'^2 + y'y - y^2\right)dx$.
    (**i**) $I[x] = \int_{t_1}^{t_2}\left(t\dot{x}^2 - 2x - x\dot{x}\right)dt$.

(**j**) $I[y] = \int_{x_1}^{x_2} \left(xy + yy' + y^2 + 2y^2y'\right) dx$.
(**k**) $I[y] = \int_{x_1}^{x_2} \sqrt{1-y'^2}\, dx$.
(**l**) $I[y] = \int_{x_1}^{x_2} \frac{yy'^2}{1+yy'}\, dx$.
Answer:
(**a**) Comparing this functional to the functional of Eq. 1, we see that $F(x,y,y') = y'^2 - 4\sqrt{xy}$ and hence the Euler-Lagrange equation (i.e. Eq. 2) is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y'^2 - 4\sqrt{xy}\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2 - 4\sqrt{xy}\right]\right) &= 0 \\
-\frac{4x}{2\sqrt{xy}} - \frac{d}{dx}(2y') &= 0 \\
-\frac{2x}{\sqrt{xy}} - 2y'' &= 0 \\
y'' + \frac{x}{\sqrt{xy}} &= 0
\end{aligned}$$

(**b**) Comparing this functional to the functional of Eq. 1 (noting that x, y, y' correspond to $t, x, \dot{x}$), we see that $F(t,x,\dot{x}) = tx^2 + \dot{x}^2$ and hence the Euler-Lagrange equation (i.e. Eq. 2) is:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[tx^2 + \dot{x}^2\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left[tx^2 + \dot{x}^2\right]\right) &= 0 \\
2tx - \frac{d}{dt}(2\dot{x}) &= 0 \\
2tx - 2\ddot{x} &= 0 \\
\ddot{x} - tx &= 0
\end{aligned}$$

(**c**) Comparing this functional to the functional of Eq. 1 (noting that x, y, y' correspond to $t, \phi, \dot{\phi}$), we see that $F\left(t, \phi, \dot{\phi}\right) = t\cos\phi + t^2\dot{\phi}$ and hence the Euler-Lagrange equation (i.e. Eq. 2) is:

$$\begin{aligned}
\frac{\partial}{\partial \phi}\left[t\cos\phi + t^2\dot{\phi}\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\phi}}\left[t\cos\phi + t^2\dot{\phi}\right]\right) &= 0 \\
-t\sin\phi - \frac{d}{dt}\left(t^2\right) &= 0 \\
t\sin\phi + 2t &= 0
\end{aligned}$$

(**d**) We have $F(x,y,y') = y'^2 - xy'^2$ and since F is independent of y we can use Eq. 4, that is:

$$\begin{aligned}
\frac{\partial}{\partial y'}\left(y'^2 - xy'^2\right) &= C \\
2y' - 2xy' &= C \\
y'(1-x) &= D \qquad\qquad (D = C/2)
\end{aligned}$$

(**e**) We have $F(t,x,\dot{x}) = t^2\dot{x} - x^2$ and hence the Euler-Lagrange equation (i.e. Eq. 2 noting that x, y, y' correspond to $t, x, \dot{x}$) is:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[t^2\dot{x} - x^2\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left[t^2\dot{x} - x^2\right]\right) &= 0 \\
-2x - \frac{d}{dt}\left(t^2\right) &= 0 \\
-2x - 2t &= 0 \\
x + t &= 0
\end{aligned}$$

(**f**) We have $F(x,y,y') = y'^2 - ye^{ax} + y^2$ and hence the Euler-Lagrange equation (i.e. Eq. 2) is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y'^2 - ye^{ax} + y^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2 - ye^{ax} + y^2\right]\right) &= 0 \\
-e^{ax} + 2y - \frac{d}{dx}(2y') &= 0 \\
-e^{ax} + 2y - 2y'' &= 0 \\
y'' - y + \frac{e^{ax}}{2} &= 0
\end{aligned}$$

(**g**) We have $F(x,y,y') = y'^3$ and hence we can use Eq. 4, that is:

$$\begin{aligned}
\frac{\partial y'^3}{\partial y'} &= C \\
3y'^2 &= C \\
y'^2 &= D \qquad (D = C/3)
\end{aligned}$$

which is consistent with footnote [12] since $y' = \text{constant}$.
(**h**) We have $F(x,y,y') = y'^2 + y'y - y^2$ and since F is independent of x we can use Eq. 3, that is:

$$\begin{aligned}
\left(y'^2 + y'y - y^2\right) - y'\frac{\partial}{\partial y'}\left(y'^2 + y'y - y^2\right) &= C \\
y'^2 + y'y - y^2 - y'(2y' + y) &= C \\
y'^2 + y'y - y^2 - 2y'^2 - y'y &= C \\
-y'^2 - y^2 &= C \\
y'^2 + y^2 &= D \qquad (D = -C)
\end{aligned}$$

Note: referring to remark 4 in the text, we note that $y'y = \frac{d(y^2/2)}{dx}$ and hence it is a total derivative of $y^2/2$. We could therefore use the suggested simplification in that remark and obtain the Euler-Lagrange equation using $F = y'^2 - y^2$, that is:

$$\begin{aligned}
\left(y'^2 - y^2\right) - y'\frac{\partial}{\partial y'}\left(y'^2 - y^2\right) &= C \\
y'^2 - y^2 - y'(2y') &= C \\
y'^2 - y^2 - 2y'^2 &= C \\
y'^2 + y^2 &= D
\end{aligned}$$

(**i**) We have $F(t,x,\dot{x}) = t\dot{x}^2 - 2x - x\dot{x}$ and hence the Euler-Lagrange equation (i.e. Eq. 2 noting that x, y, y' correspond to $t, x, \dot{x}$) is:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[t\dot{x}^2 - 2x - x\dot{x}\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left[t\dot{x}^2 - 2x - x\dot{x}\right]\right) &= 0 \\
-2 - \dot{x} - \frac{d}{dt}(2t\dot{x} - x) &= 0 \\
-2 - \dot{x} - (2\dot{x} + 2t\ddot{x} - \dot{x}) &= 0 \\
-2 - \dot{x} - 2\dot{x} - 2t\ddot{x} + \dot{x} &= 0 \\
-2 - 2\dot{x} - 2t\ddot{x} &= 0 \\
t\ddot{x} + \dot{x} + 1 &= 0
\end{aligned}$$

Note: referring to remark 4 in the text, we note that $x\dot{x} = \frac{d(x^2/2)}{dt}$ and hence it is a total derivative of $x^2/2$. We could therefore use the suggested simplification in that remark and obtain the Euler-Lagrange equation using $F = t\dot{x}^2 - 2x$, that is:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[t\dot{x}^2 - 2x\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left[t\dot{x}^2 - 2x\right]\right) &= 0\\
-2 - \frac{d}{dt}\left(2t\dot{x}\right) &= 0\\
-2 - \left(2\dot{x} + 2t\ddot{x}\right) &= 0\\
-2 - 2\dot{x} - 2t\ddot{x} &= 0\\
t\ddot{x} + \dot{x} + 1 &= 0
\end{aligned}$$

(**j**) We have $F(x, y, y') = xy + yy' + y^2 + 2y^2y'$ and hence the Euler-Lagrange equation (i.e. Eq. 2) is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[xy + yy' + y^2 + 2y^2y'\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[xy + yy' + y^2 + 2y^2y'\right]\right) &= 0\\
x + y' + 2y + 4yy' - \frac{d}{dx}\left(y + 2y^2\right) &= 0\\
x + y' + 2y + 4yy' - y' - 4yy' &= 0\\
x + 2y &= 0\\
y &= -\frac{x}{2}
\end{aligned}$$

Note: referring to remark 4 in the text, we note that yy' is a total derivative of $y^2/2$ and $2y^2y'$ is a total derivative of $\frac{2}{3}y^3$. We could therefore use the suggested simplification in that remark and obtain the Euler-Lagrange equation using $F = xy + y^2$, that is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[xy + y^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[xy + y^2\right]\right) &= 0\\
x + 2y - \frac{d}{dx}(0) &= 0\\
x + 2y &= 0\\
y &= -\frac{x}{2}
\end{aligned}$$

(**k**) We have $F(x, y, y') = \sqrt{1 - y'^2}$ and hence the Euler-Lagrange equation (i.e. Eq. 4 because y does not appear in F) is:

$$\begin{aligned}
\frac{\partial}{\partial y'}\left[\sqrt{1 - y'^2}\right] &= C\\
\frac{-y'}{\sqrt{1 - y'^2}} &= C\\
y'^2 &= \frac{C^2}{1 + C^2}
\end{aligned}$$

which is consistent with footnote [12] since $y' = \text{constant}$.

(**l**) We have $F(x, y, y') = \frac{yy'^2}{1+yy'}$ and since F is independent of x we can use Eq. 3, that is:

$$\begin{aligned}
\left(\frac{yy'^2}{1 + yy'}\right) - y'\frac{\partial}{\partial y'}\left(\frac{yy'^2}{1 + yy'}\right) &= C\\
\frac{yy'^2}{1 + yy'} - y'\left(\frac{2yy'}{1 + yy'} - \frac{yy'^2\,(y)}{\left(1 + yy'\right)^2}\right) &= C\\
\frac{yy'^2}{1 + yy'} - \frac{2yy'^2}{1 + yy'} + \frac{y^2y'^3}{\left(1 + yy'\right)^2} &= C
\end{aligned}$$

$$\begin{aligned}
-\frac{yy'^2}{1+yy'}+\frac{y^2y'^3}{(1+yy')^2} &= C\\
-\frac{yy'^2(1+yy')}{(1+yy')^2}+\frac{y^2y'^3}{(1+yy')^2} &= C\\
-\frac{yy'^2}{(1+yy')^2} &= C\\
yy'^2 &= D(1+yy')^2 \qquad\qquad (D=-C)
\end{aligned}$$

11. Find the extremizing (or stationarizing) functions of the following functional integrals and verify the results:
(**a**) $I[y]=\int_{x_1}^{x_2}x^3y'^2\,dx$.
(**b**) $I[x]=\int_{t_1}^{t_2}\left(x^2+\dot{x}^2\right)dt$.
(**c**) $I[y]=\int_{x_1}^{x_2}x\sqrt{1-y'^2}\,dx$.
(**d**) $I[y]=\int_{x_1}^{x_2}\left(y'^2-xy\right)dx$.
(**e**) $I[r]=\int_{\theta_1}^{\theta_2}\left(\frac{1}{2}r'^2-\cos\theta\right)d\theta$ (where the prime means $d/d\theta$).
(**f**) $I[y]=\int_{x_1}^{x_2}\left(y'^2-axy'\right)dx$ (where a is a constant).
(**g**) $I[y]=\int_{x_1}^{x_2}\left(y'^{1/2}-x^{1/2}\right)dx$.
(**h**) $I[y]=\int_{x_1}^{x_2}\left(y'^2-ky\right)dx$ (where k is a constant).
(**i**) $I[y]=\int_{x_1}^{x_2}\left(y'^2-2y+5x\right)dx$.
(**j**) $I[y]=\int_{x_1}^{x_2}\left(y'^2+2yy'-y^2\right)dx$.
(**k**) $I[y]=\int_{x_1}^{x_2}\left(y'^2+2yy'+4y^2\right)dx$.
(**l**) $I[y]=\int_{x_1}^{x_2}\sqrt{x\left(1+y'^2\right)}\,dx$.
Answer:
(**a**) Comparing this functional to the functional of Eq. 1, we see that $F=x^3y'^2$. Now, since F is independent of y we can use Eq. 4, that is:

$$\begin{aligned}
\frac{\partial}{\partial y'}\left(x^3y'^2\right) &= C\\
2x^3y' &= C
\end{aligned}$$

This is the Euler-Lagrange equation[17] which we solve as follows:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{C}{2x^3}\\
y &= -\frac{C}{4x^2}+D \qquad\qquad (D\text{ is a consatnt})
\end{aligned}$$

which is the extremizing function.
To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned}
2x^3\left(-\frac{C}{4x^2}+D\right)' &\overset{?}{=} C\\
2x^3\left(-\frac{-2C}{4x^3}+0\right) &\overset{?}{=} C\\
2x^3\left(\frac{C}{2x^3}\right) &\overset{?}{=} C\\
C &= C
\end{aligned}$$

---

[17] When we say "This is the Euler-Lagrange equation" here and in similar contexts, we mean "for the specific problem". So in fact, it is an instantiation of the Euler-Lagrange equation in its general form (as given by Eq. 2 and its variant forms).

(**b**) Comparing this functional to the functional of Eq. 1 (noting that x, y, y' correspond to $t, x, \dot{x}$), we see that $F = x^2 + \dot{x}^2$. Now, since F is independent of t we can use Eq. 3, that is:

$$\begin{aligned}
\left(x^2+\dot{x}^2\right) - \dot{x}\frac{\partial}{\partial \dot{x}}\left(x^2+\dot{x}^2\right) &= C\\
x^2+\dot{x}^2-\dot{x}\left(2\dot{x}\right) &= C\\
x^2+\dot{x}^2-2\dot{x}^2 &= C\\
x^2-\dot{x}^2 &= C\\
\dot{x}^2 &= x^2-C
\end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned}
\dot{x} &= \pm\sqrt{x^2-C}\\
\frac{dx}{dt} &= \pm\sqrt{x^2-C}\\
\frac{dx}{\sqrt{x^2-C}} &= \pm dt\\
\ln\left(\sqrt{x^2-C}+x\right) &= \pm t + D && \text{(integrating both sides)}\\
\sqrt{x^2-C}+x &= e^{\pm t+D}\\
\sqrt{x^2-C}+x &= Ee^{\pm t} && (E = e^D)
\end{aligned}$$

So, the extremizing function $x(t)$ is given implicitly by the last equation.
To verify the result we show that the last equation is equivalent to the Euler-Lagrange equation, that is:

$$\begin{aligned}
\sqrt{x^2-C}+x &= Ee^{\pm t}\\
\ln\left(\sqrt{x^2-C}+x\right) &= \ln E \pm t && \text{(taking natural logarithm of both sides)}\\
\frac{\left(\frac{x}{\sqrt{x^2-C}}+1\right)}{\sqrt{x^2-C}+x}\frac{dx}{dt} &= \pm 1 && \text{(taking derivative of both sides with respect to } t\text{)}\\
\frac{\frac{x+\sqrt{x^2-C}}{\sqrt{x^2-C}}}{\sqrt{x^2-C}+x}\frac{dx}{dt} &= \pm 1\\
\frac{1}{\sqrt{x^2-C}}\dot{x} &= \pm 1\\
\frac{1}{x^2-C}\dot{x}^2 &= 1\\
\dot{x}^2 &= x^2-C
\end{aligned}$$

which is the Euler-Lagrange equation.
(**c**) We have $F(x, y, y') = x\sqrt{1-y'^2}$, and since it is independent of y we can use Eq. 4, that is:

$$\begin{aligned}
\frac{\partial}{\partial y'}\left(x\sqrt{1-y'^2}\right) &= C\\
\frac{xy'}{\sqrt{1-y'^2}} &= -C
\end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$x^2y'^2 = C^2 - C^2y'^2$$

$$\begin{aligned} y'^2 &= \frac{C^2}{x^2+C^2} \\ y' &= \frac{\pm 1}{\sqrt{(x/C)^2+1}} \\ y &= \pm C \operatorname{arcsinh}\left(\frac{x}{C}\right) + D \end{aligned}$$

which is the extremizing function.
To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned} \frac{x\left[\pm C\operatorname{arcsinh}\left(\frac{x}{C}\right)+D\right]'}{\sqrt{1-\left[\pm C\operatorname{arcsinh}\left(\frac{x}{C}\right)+D\right]'^2}} &\overset{?}{=} -C \\ \frac{x\left[\frac{\pm 1}{\sqrt{(x/C)^2+1}}\right]}{\sqrt{1-\left[\frac{\pm 1}{\sqrt{(x/C)^2+1}}\right]^2}} &\overset{?}{=} -C \\ \frac{x^2\left[\frac{1}{(x/C)^2+1}\right]}{1-\left[\frac{\pm 1}{\sqrt{(x/C)^2+1}}\right]^2} &\overset{?}{=} C^2 \\ \frac{x^2\left[\frac{1}{(x/C)^2+1}\right]}{1-\left[\frac{1}{(x/C)^2+1}\right]} &\overset{?}{=} C^2 \\ \frac{x^2\left[\frac{1}{(x/C)^2+1}\right]}{\frac{(x/C)^2+1-1}{(x/C)^2+1}} &\overset{?}{=} C^2 \\ \frac{x^2}{(x/C)^2} &\overset{?}{=} C^2 \\ C^2 &= C^2 \end{aligned}$$

(**d**) We have $F(x, y, y') = y'^2 - xy$ and hence the Euler-Lagrange equation (Eq. 2) is:

$$\begin{aligned} \frac{\partial}{\partial y}\left[y'^2 - xy\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2 - xy\right]\right) &= 0 \\ -x - \frac{d}{dx}(2y') &= 0 \\ -x - 2y'' &= 0 \\ 2y'' + x &= 0 \end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned} y'' &= -\frac{1}{2}x \\ y' &= -\frac{1}{4}x^2 + C \\ y &= -\frac{1}{12}x^3 + Cx + D \end{aligned}$$

which is the extremizing function.

To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned}2\left(-\frac{1}{12}x^3+Cx+D\right)''+x &\overset{?}{=} 0\\ 2\left(-\frac{1}{4}x^2+C+0\right)'+x &\overset{?}{=} 0\\ 2\left(-\frac{1}{2}x+0\right)+x &\overset{?}{=} 0\\ -x+x &= 0\end{aligned}$$

(**e**) We have $F(\theta,r,r')=\frac{1}{2}r'^2-\cos\theta$ and since F is independent of r we use Eq. 4 (noting that x,y,y' correspond to θ,r,r' and the prime means $d/d\theta$), that is:

$$\begin{aligned}\frac{\partial}{\partial r'}\left(\frac{1}{2}r'^2-\cos\theta\right) &= C\\ r' &= C\end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned}\frac{dr}{d\theta} &= C\\ r &= C\theta+D\end{aligned}$$

which is the extremizing function.
To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned}(C\theta+D)' &\overset{?}{=} C\\ (C+0) &\overset{?}{=} C\\ C &= C\end{aligned}$$

(**f**) We have $F(x,y,y')=y'^2-axy'$ and since F is independent of y we can use Eq. 4, that is:

$$\begin{aligned}\frac{\partial}{\partial y'}\left(y'^2-axy'\right) &= C\\ 2y'-ax &= C\end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned}y' &= \frac{a}{2}x+\frac{C}{2}\\ y &= \frac{a}{4}x^2+\frac{C}{2}x+D\end{aligned}$$

which is the extremizing function.
To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned}2\left(\frac{a}{4}x^2+\frac{C}{2}x+D\right)'-ax &\overset{?}{=} C\\ 2\left(\frac{a}{2}x+\frac{C}{2}+0\right)-ax &\overset{?}{=} C\\ ax+C-ax &\overset{?}{=} C\\ C &= C\end{aligned}$$

(**g**) We have $F(x,y,y') = y'^{1/2} - x^{1/2}$ and since F is independent of y we can use Eq. 4, that is:

$$\begin{aligned}
\frac{\partial}{\partial y'}\left(y'^{1/2} - x^{1/2}\right) &= C \\
\frac{1}{2}y'^{-1/2} &= C
\end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned}
y'^{-1/2} &= 2C \\
y'^{-1} &= 4C^2 \\
y' &= D \qquad (D = 1/[4C^2]) \\
y &= Dx + E
\end{aligned}$$

which is the extremizing function.
To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned}
\frac{1}{2}\left(Dx+E\right)'^{-1/2} &\overset{?}{=} C \\
\frac{1}{2}\left(D+0\right)^{-1/2} &\overset{?}{=} C \\
\frac{1}{2}D^{-1/2} &\overset{?}{=} C \\
\frac{1}{2}2C &\overset{?}{=} C \\
C &= C
\end{aligned}$$

(**h**) We have $F(x,y,y') = y'^2 - ky$ and since F is independent of x we can use Eq. 3, that is:

$$\begin{aligned}
\left(y'^2 - ky\right) - y'\frac{\partial}{\partial y'}\left(y'^2 - ky\right) &= C \\
y'^2 - ky - y'\left(2y'\right) &= C \\
-ky - y'^2 &= C \\
ky + y'^2 &= D \qquad (D = -C)
\end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned}
y' &= \pm\sqrt{D - ky} \\
\pm\frac{dy}{\sqrt{D-ky}} &= dx \\
\mp\frac{2}{k}\sqrt{D-ky} &= x + E \\
y &= \frac{1}{k}\left[D - \frac{k^2}{4}\left(x+E\right)^2\right]
\end{aligned}$$

which is the extremizing function.
To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned}
k\left(\frac{1}{k}\left[D - \frac{k^2}{4}\left(x+E\right)^2\right]\right) + \left(\frac{1}{k}\left[D - \frac{k^2}{4}\left(x+E\right)^2\right]\right)'^2 &\overset{?}{=} D \\
D - \frac{k^2}{4}\left(x+E\right)^2 + \left(\frac{1}{k}\left[0 - \frac{k^2}{2}\left(x+E\right)\right]\right)^2 &\overset{?}{=} D
\end{aligned}$$

$$\begin{aligned}
D-\frac{k^2}{4}(x+E)^2+\left(-\frac{k}{2}(x+E)\right)^2 &\overset{?}{=} D\\
D-\frac{k^2}{4}(x+E)^2+\frac{k^2}{4}(x+E)^2 &\overset{?}{=} D\\
D &= D
\end{aligned}$$

(**i**) We have $F(x,y,y')=y'^2-2y+5x$ and hence the Euler-Lagrange equation (i.e. Eq. 2) is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y'^2-2y+5x\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2-2y+5x\right]\right) &= 0\\
-2-\frac{d}{dx}(2y') &= 0\\
y''+1 &= 0
\end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned}
y'' &= -1\\
y' &= -x+C\\
y &= -\frac{x^2}{2}+Cx+D
\end{aligned}$$

which is the extremizing function.
To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned}
\left(-\frac{x^2}{2}+Cx+D\right)''+1 &\overset{?}{=} 0\\
(-x+C+0)'+1 &\overset{?}{=} 0\\
(-1+0)+1 &\overset{?}{=} 0\\
-1+1 &= 0
\end{aligned}$$

(**j**) We have $F(x,y,y')=y'^2+2yy'-y^2$ and hence the Euler-Lagrange equation (i.e. Eq. 2) is:[18]

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y'^2+2yy'-y^2\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2+2yy'-y^2\right]\right) &= 0\\
2y'-2y-\frac{d}{dx}(2y'+2y) &= 0\\
2y'-2y-2y''-2y' &= 0\\
y''+y &= 0
\end{aligned}$$

This is the Euler-Lagrange equation whose solution is:

$$y=a\sin x+b\cos x \qquad (a \text{ and } b \text{ are constants})$$

which is the extremizing function.
To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned}
(a\sin x+b\cos x)''+(a\sin x+b\cos x) &\overset{?}{=} 0\\
(a\cos x-b\sin x)'+(a\sin x+b\cos x) &\overset{?}{=} 0
\end{aligned}$$

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[18] Referring to remark 4 in the text and noting that $2yy'$ is a total derivative of y^2 we can obtain the Euler-Lagrange equation using $F=y'^2-y^2$.

$$\begin{aligned}(-a\sin x - b\cos x) + (a\sin x + b\cos x) &\overset{?}{=} 0\\ 0 &= 0\end{aligned}$$

(**k**) We have $F(x,y,y') = y'^2 + 2yy' + 4y^2$ and hence the Euler-Lagrange equation (i.e. Eq. 2) is:[19]

$$\begin{aligned}\frac{\partial}{\partial y}\left[y'^2 + 2yy' + 4y^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2 + 2yy' + 4y^2\right]\right) &= 0\\ 2y' + 8y - \frac{d}{dx}(2y' + 2y) &= 0\\ 2y' + 8y - 2y'' - 2y' &= 0\\ y'' - 4y &= 0\end{aligned}$$

This is the Euler-Lagrange equation whose solution is:

$$y = a\sinh(2x) + b\cosh(2x) \qquad\qquad (a \text{ and } b \text{ are constants})$$

which is the extremizing function.

To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:

$$\begin{aligned}\big[a\sinh(2x) + b\cosh(2x)\big]'' - 4\big[a\sinh(2x) + b\cosh(2x)\big] &\overset{?}{=} 0\\ \big[2a\cosh(2x) + 2b\sinh(2x)\big]' - 4\big[a\sinh(2x) + b\cosh(2x)\big] &\overset{?}{=} 0\\ \big[4a\sinh(2x) + 4b\cosh(2x)\big] - 4\big[a\sinh(2x) + b\cosh(2x)\big] &\overset{?}{=} 0\\ 0 &= 0\end{aligned}$$

(**l**) We have $F(x,y,y') = \sqrt{x\,(1+y'^2)}$ and hence the Euler-Lagrange equation (i.e. Eq. 4 since F has no explicit dependency on y) is:

$$\begin{aligned}\frac{\partial\sqrt{x\,(1+y'^2)}}{\partial y'} &= C\\ \frac{y'\sqrt{x}}{\sqrt{1+y'^2}} &= C\\ y'^2 x &= C^2\left(1+y'^2\right)\\ y'^2 &= \frac{C^2}{x-C^2}\\ y' &= \pm\sqrt{\frac{C^2}{x-C^2}} \qquad (10)\end{aligned}$$

This is the Euler-Lagrange equation whose solution is:

$$y = \pm 2\sqrt{C^2\,(x-C^2)} + D$$

which is the extremizing function.

To verify the result we substitute from the last equation into the Euler-Lagrange equation, that is:[20]

$$\begin{aligned}\left[\pm 2\sqrt{C^2\,(x-C^2)} + D\right]' &\overset{?}{=} \pm\sqrt{\frac{C^2}{x-C^2}}\\ \pm\frac{2\sqrt{C^2}}{2\sqrt{x-C^2}} + 0 &\overset{?}{=} \pm\sqrt{\frac{C^2}{x-C^2}}\\ \pm\sqrt{\frac{C^2}{x-C^2}} &= \pm\sqrt{\frac{C^2}{x-C^2}}\end{aligned}$$

---

[19] Referring to remark 4 in the text and noting that $2yy'$ is a total derivative of y^2 we can obtain the Euler-Lagrange equation using $F = y'^2 + 4y^2$.

[20] In fact, the result can be verified more easily by just differentiating y with respect to x to obtain Eq. 10.

12. Find the extremizing (or stationarizing) functions of the following functional integrals as well as the specific solutions for the given boundary conditions:
(a) $I[y] = \int_{x_1}^{x_2} \left(y'^2 + ky^2\right) dx$ with $y(x_1 = 0) = 0$ and $y(x_2 = 1) = 1$ ($k > 0$ is a constant).
(b) $I[y] = \int_{x_1}^{x_2} \left(xy - y'^2\right) dx$ with $y(x_1 = 0) = 0$ and $y(x_2 = 1) = 12$.
(c) $I[y] = \int_{x_1}^{x_2} \frac{y'^2}{x^3} dx$ with $y(x_1 = 2) = 1$ and $y(x_2 = 4) = 31$.
(d) $I[y] = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x} dx$ with $y(x_1 = 0) = \sqrt{2}$ and $y(x_2 = 1) = 1$.
(e) $I[y] = \int_{x_1}^{x_2} \left(y'^2 - y^2 + y\cosh x\right) dx$ with $y(x_1 = 0) = 0$ and $y(x_2 = \pi/2) = 1$.
(f) $I[y] = \int_{x_1}^{x_2} \left(y'^2 + y^2 - 4y\cos x\right) dx$ with $y(x_1 = 0) = 1$ and $y(x_2 = \pi) = 1$.
(g) $I[y] = \int_{x_1}^{x_2} \left(y'^2 - y^2 - 2xy\right) dx$ with $y(x_1 = 0) = 1$ and $y(x_2 = 1) = 2$.
(h) $I[y] = \int_{x_1}^{x_2} \left(y^2 - x^2 y\right) dx$ with $y(x_1 = 0) = 0$ and $y(x_2 = 1) = 3$.
Answer:
(a) Comparing this functional to the functional of Eq. 1, we see that $F = y'^2 + ky^2$ and hence from the Euler-Lagrange equation (Eq. 2) we have:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y'^2 + ky^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2 + ky^2\right]\right) &= 0 \\
2ky - \frac{d}{dx}(2y') &= 0 \\
2ky - 2y'' &= 0 \\
y'' - ky &= 0
\end{aligned}$$

So, the solution is $y = C\cosh(\sqrt{k}x) + D\sinh(\sqrt{k}x)$ which can be checked by substitution into the Euler-Lagrange equation, that is:

$$\begin{aligned}
\left[C\cosh\left(\sqrt{k}x\right) + D\sinh\left(\sqrt{k}x\right)\right]'' - k\left[C\cosh\left(\sqrt{k}x\right) + D\sinh\left(\sqrt{k}x\right)\right] &\overset{?}{=} 0 \\
\left[\sqrt{k}C\sinh\left(\sqrt{k}x\right) + \sqrt{k}D\cosh\left(\sqrt{k}x\right)\right]' - k\left[C\cosh\left(\sqrt{k}x\right) + D\sinh\left(\sqrt{k}x\right)\right] &\overset{?}{=} 0 \\
\left[kC\cosh\left(\sqrt{k}x\right) + kD\sinh\left(\sqrt{k}x\right)\right] - k\left[C\cosh\left(\sqrt{k}x\right) + D\sinh\left(\sqrt{k}x\right)\right] &\overset{?}{=} 0 \\
0 &= 0
\end{aligned}$$

Now, from the condition $y(x_1 = 0) = 0$ we get $C = 0$ while from the condition $y(x_2 = 1) = 1$ we get $D = 1/\sinh(\sqrt{k})$ and hence the specific solution is:

$$y = \frac{\sinh\left(\sqrt{k}x\right)}{\sinh\left(\sqrt{k}\right)}$$

Note: this solution (with $k = 1$) is plotted later in the book (see Figures 69, 77 and 78).
(b) We have $F = xy - y'^2$ and hence from the Euler-Lagrange equation (Eq. 2) we have:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[xy - y'^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[xy - y'^2\right]\right) &= 0 \\
x - \frac{d}{dx}(-2y') &= 0 \\
x + 2y'' &= 0 \\
\frac{d^2y}{dx^2} &= -\frac{x}{2} \\
\frac{dy}{dx} &= -\frac{x^2}{4} + C
\end{aligned}$$

$$y = -\frac{x^3}{12} + Cx + D \qquad (C \text{ and } D \text{ are constants})$$

Now, from the condition $y(x_1 = 0) = 0$ we get $D = 0$ while from the condition $y(x_2 = 1) = 12$ we get $C = 145/12$ and hence the specific solution is:

$$y = \frac{-x^3 + 145x}{12}$$

(**c**) We have $F = y'^2/x^3$. Now, since F is independent of y we can use Eq. 4, that is:

$$\begin{aligned}
\frac{\partial}{\partial y'}\left[\frac{y'^2}{x^3}\right] &= C \\
\frac{2y'}{x^3} &= C \\
\frac{dy}{dx} &= \frac{C}{2}x^3 \\
y &= \frac{C}{8}x^4 + D
\end{aligned}$$

Now, from the condition $y(x_1 = 2) = 1$ we get $2C + D = 1$ while from the condition $y(x_2 = 4) = 31$ we get $32C + D = 31$ and hence $C = 1$ and $D = -1$. So, the specific solution is:

$$y = \frac{x^4}{8} - 1$$

Note: this solution is plotted later in the book (see Figures 71 and 76).

(**d**) We have $F = \frac{\sqrt{1+y'^2}}{x}$ and because it does not contain y we can use Eq. 4, that is:

$$\begin{aligned}
\frac{\partial}{\partial y'}\left[\frac{\sqrt{1+y'^2}}{x}\right] &= C \\
\frac{y'}{x\sqrt{1+y'^2}} &= C \\
y'^2 &= C^2x^2\left(1+y'^2\right) \\
y'^2 &= \frac{C^2x^2}{1-C^2x^2} \\
y' &= \pm\frac{Cx}{\sqrt{1-C^2x^2}} \\
y &= \mp\frac{1}{C}\sqrt{1-C^2x^2} + D \\
x^2 + (y-D)^2 &= \frac{1}{C^2}
\end{aligned}$$

Now, from the condition $y(x_1 = 0) = \sqrt{2}$ we get $\left(\sqrt{2} - D\right)^2 = \frac{1}{C^2}$ while from the condition $y(x_2 = 1) = 1$ we get $1 + (1-D)^2 = \frac{1}{C^2}$ and hence $C = 1/\sqrt{2}$ and $D = 0$. So, the specific solution is:

$$x^2 + y^2 = 2$$

(**e**) We have $F = y'^2 - y^2 + y\cosh x$ and hence from the Euler-Lagrange equation (Eq. 2) we have:

$$\frac{\partial}{\partial y}\left[y'^2 - y^2 + y\cosh x\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2 - y^2 + y\cosh x\right]\right) = 0$$

$$\begin{aligned}-2y+\cosh x-\frac{d}{dx}\left(2y'\right) &= 0\\ -2y+\cosh x-2y'' &= 0\\ y''+y-\frac{1}{2}\cosh x &= 0\end{aligned}$$

So, the solution is obviously a combination of sinusoidal and hyperbolic functions (i.e. $y=C\cos x+D\sin x+\frac{1}{4}\cosh x$) which can be checked by substitution into the Euler-Lagrange equation, that is:

$$\begin{aligned}\left[C\cos x+D\sin x+\frac{1}{4}\cosh x\right]''+\left[C\cos x+D\sin x+\frac{1}{4}\cosh x\right]-\frac{1}{2}\cosh x &\overset{?}{=} 0\\ \left[-C\sin x+D\cos x+\frac{1}{4}\sinh x\right]'+\left[C\cos x+D\sin x+\frac{1}{4}\cosh x\right]-\frac{1}{2}\cosh x &\overset{?}{=} 0\\ \left[-C\cos x-D\sin x+\frac{1}{4}\cosh x\right]+\left[C\cos x+D\sin x+\frac{1}{4}\cosh x\right]-\frac{1}{2}\cosh x &\overset{?}{=} 0\\ 0 &= 0\end{aligned}$$

Now, from the condition $y\left(x_1=0\right)=0$ we get $C+\frac{1}{4}=0$ while from the condition $y\left(x_2=\pi/2\right)=1$ we get $D+\frac{1}{4}\cosh\frac{\pi}{2}=1$ and hence $C=-\frac{1}{4}$ and $D=1-\frac{1}{4}\cosh\frac{\pi}{2}$. So, the specific solution is:

$$y=-\frac{1}{4}\cos x+\left(1-\frac{1}{4}\cosh\frac{\pi}{2}\right)\sin x+\frac{1}{4}\cosh x$$

Note: this solution is plotted later in the book (see Figure 72).
(**f**) We have $F=y'^2+y^2-4y\cos x$ and hence from the Euler-Lagrange equation (Eq. 2) we have:

$$\begin{aligned}\frac{\partial}{\partial y}\left[y'^2+y^2-4y\cos x\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2+y^2-4y\cos x\right]\right) &= 0\\ 2y-4\cos x-\frac{d}{dx}\left(2y'\right) &= 0\\ 2y-4\cos x-2y'' &= 0\\ y''-y+2\cos x &= 0\end{aligned}$$

So, the solution is obviously a combination of hyperbolic and sinusoidal functions (i.e. $y=C\cosh x+D\sinh x+\cos x$) which can be checked by substitution into the Euler-Lagrange equation, that is:

$$\begin{aligned}\left[C\cosh x+D\sinh x+\cos x\right]''-\left[C\cosh x+D\sinh x+\cos x\right]+2\cos x &\overset{?}{=} 0\\ \left[C\sinh x+D\cosh x-\sin x\right]'-\left[C\cosh x+D\sinh x+\cos x\right]+2\cos x &\overset{?}{=} 0\\ \left[C\cosh x+D\sinh x-\cos x\right]-\left[C\cosh x+D\sinh x+\cos x\right]+2\cos x &\overset{?}{=} 0\\ 0 &= 0\end{aligned}$$

Now, from the condition $y\left(x_1=0\right)=1$ we get $C+1=1$ while from the condition $y\left(x_2=\pi\right)=1$ we get $C\cosh\pi+D\sinh\pi-1=1$ and hence $C=0$ and $D=2/\sinh\pi$. So, the specific solution is:

$$y=\left(\frac{2}{\sinh\pi}\right)\sinh x+\cos x$$

Note: this solution is plotted later in the book (see Figure 80).
(**g**) We have $F=y'^2-y^2-2xy$ and hence from the Euler-Lagrange equation (Eq. 2) we have:

$$\frac{\partial}{\partial y}\left[y'^2-y^2-2xy\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2-y^2-2xy\right]\right) = 0$$

$$\begin{aligned}-2y-2x-\frac{d}{dx}\left(2y'\right) &= 0\\ -2y-2x-2y'' &= 0\\ y''+y+x &= 0\end{aligned}$$

So, the solution is obviously a combination of sinusoidal and polynomial functions (i.e. $y = C\cos x + D\sin x - x$) which can be checked by substitution, that is:

$$\begin{aligned}\big[C\cos x + D\sin x - x\big]'' + \big[C\cos x + D\sin x - x\big] + x &\overset{?}{=} 0\\ \big[-C\sin x + D\cos x - 1\big]' + \big[C\cos x + D\sin x - x\big] + x &\overset{?}{=} 0\\ \big[-C\cos x - D\sin x - 0\big] + \big[C\cos x + D\sin x - x\big] + x &\overset{?}{=} 0\\ 0 &= 0\end{aligned}$$

Now, from the condition $y\left(x_1 = 0\right) = 1$ we get $C + D\times 0 - 0 = 1$ while from the condition $y\left(x_2 = 1\right) = 2$ we get $C\cos 1 + D\sin 1 - 1 = 2$ and hence $C = 1$ and $D = (3-\cos 1)/\sin 1$. So, the specific solution is:

$$y = \cos x + \left(\frac{3-\cos 1}{\sin 1}\right)\sin x - x$$

Note: this solution is plotted later in the book (see Figures 70 and 79).

(**h**) We have $F = y^2 - x^2 y$ and hence from the Euler-Lagrange equation (Eq. 2) we have:

$$\begin{aligned}\frac{\partial}{\partial y}\left[y^2 - x^2 y\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y^2 - x^2 y\right]\right) &= 0\\ 2y - x^2 - \frac{d}{dx}(0) &= 0\\ 2y - x^2 &= 0\\ y &= \frac{x^2}{2}\end{aligned}$$

Now, the first boundary condition $y\left(x_1 = 0\right) = 0$ is satisfied by this solution but the second boundary condition $y\left(x_2 = 1\right) = 3$ is not. Hence, this problem has no solution (i.e. specific solution for the given boundary conditions).

13. Find the Euler-Lagrange equation that associates the functional $I[\theta] = \int_{t_1}^{t_2}\left(\dot{\theta}^2 - \beta\theta^2\right)dt$ (with β being a constant) and investigate its solution.
Answer: We have $F(t,\theta,\dot{\theta}) = \dot{\theta}^2 - \beta\theta^2$ and hence the Euler-Lagrange equation (i.e. Eq. 2 noting that x, y, y' correspond to $t, \theta, \dot{\theta}$) is:

$$\begin{aligned}\frac{\partial}{\partial\theta}\left[\dot{\theta}^2 - \beta\theta^2\right] - \frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}\left[\dot{\theta}^2 - \beta\theta^2\right]\right) &= 0\\ -2\beta\theta - \frac{d}{dt}\left(2\dot{\theta}\right) &= 0\\ -2\beta\theta - 2\ddot{\theta} &= 0\\ \ddot{\theta} + \beta\theta &= 0\end{aligned}$$

Regarding its solution, we have three cases:

(**a**) If $\beta < 0$ then we have $\ddot{\theta} - |\beta|\,\theta = 0$ whose solution is obviously hyperbolic of the form $\theta = a\cosh\left(\sqrt{|\beta|}\,t\right) + b\sinh\left(\sqrt{|\beta|}\,t\right)$ (with a and b being constants).

(**b**) If $\beta = 0$ then we have $\ddot{\theta} = 0$ whose solution is obviously linear polynomial of the form $\theta = at + b$ (with a and b being constants).

(**c**) If $\beta > 0$ then we have $\ddot{\theta} + |\beta|\,\theta = 0$ whose solution is obviously sinusoidal of the form $\theta = a\cos\left(\sqrt{|\beta|}\,t\right) + b\sin\left(\sqrt{|\beta|}\,t\right)$ (with a and b being constants).

1.5 Variational Problems with Higher Derivatives

In some variational problems the functional integral I depends also on derivatives of the extremizing function y of higher orders. In such cases, more terms should be added to the Euler-Lagrange equation to account for this extra dependency. For example, if I depends on the second order derivative y'' as well and hence:

$$I\left[y\right]=\int_{x_1}^{x_2} F\left(x,y,y',y''\right)\,dx \tag{11}$$

$\big[$where the values of y and y' at the end points are given by the conditions $y(x_1)=C_1$, $y(x_2)=C_2$, $y'(x_1)=C_3$ and $y'(x_2)=C_4\big]$ then the Euler-Lagrange equation[21] becomes:

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=0 \tag{12}$$

This can be generalized if I depends on derivatives higher than the second (up to the n^{th} derivative) as well, that is:

$$I\left[y\right]=\int_{x_1}^{x_2} F\left(x,y,y^{(1)},\cdots,y^{(n)}\right)\,dx \qquad \text{and} \qquad \frac{\partial F}{\partial y}+\sum_{i=1}^{n}(-1)^i\,\frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y^{(i)}}\right)=0 \tag{13}$$

where $y^{(1)},y^{(n)},y^{(i)}$ are the $1^{st},n^{th},i^{th}$ derivatives of y, i.e. $y^{(1)}=dy/dx$, $y^{(n)}=d^ny/dx^n$ and $y^{(i)}=d^iy/dx^i$.

Problems

1. What is the Euler-Lagrange equation when:
   (**a**) $F(x,y,y',y'')=yy''+xy'^2$.
   (**b**) $F\left(t,\theta,\dot{\theta},\ddot{\theta}\right)=2\sin\theta+\dot{\theta}^2+3\ddot{\theta}$.
   (**c**) $F(x,y,y',y'')=ay''-y'^2+cxy$ (with a and c being constants).
   (**d**) $F(s,\phi,\phi',\phi'')=\phi''^2+a\phi'-bs^2\phi+cs^3$ (with a,b,c being constants and the prime means d/ds).

   Answer:

   (**a**) Using Eq. 12 we have:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[yy''+xy'^2\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[yy''+xy'^2\right]\right)+\frac{d^2}{dx^2}\left(\frac{\partial}{\partial y''}\left[yy''+xy'^2\right]\right) &= 0\\
y''-\frac{d}{dx}\left(2xy'\right)+\frac{d^2}{dx^2}\left(y\right) &= 0\\
y''-\left(2y'+2xy''\right)+y'' &= 0\\
2y''-2y'-2xy'' &= 0\\
y''\left(1-x\right)-y' &= 0
\end{aligned}$$

   (**b**) Using Eq. 12 (with x,y,y',y'' corresponding to $t,\theta,\dot{\theta},\ddot{\theta}$) we have:

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left[2\sin\theta+\dot{\theta}^2+3\ddot{\theta}\right]-\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}\left[2\sin\theta+\dot{\theta}^2+3\ddot{\theta}\right]\right)+\frac{d^2}{dt^2}\left(\frac{\partial}{\partial\ddot{\theta}}\left[2\sin\theta+\dot{\theta}^2+3\ddot{\theta}\right]\right) &= 0\\
2\cos\theta-\frac{d}{dt}\left(2\dot{\theta}\right)+\frac{d^2}{dt^2}\left(3\right) &= 0\\
2\cos\theta-2\ddot{\theta}+0 &= 0\\
\ddot{\theta}-\cos\theta &= 0
\end{aligned}$$

---

[21] Although we call this equation (and its alike) the Euler-Lagrange equation it is not really the Euler-Lagrange equation but an extension (or generalization) of it. In fact, in some texts this equation (and its alike) is labeled differently (e.g. Euler-Poisson) but we keep our labeling (i.e. Euler-Lagrange equation) for simplicity and to avoid possible confusion.

(**c**) Using Eq. 12 we have:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[ay''-y'^2+cxy\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[ay''-y'^2+cxy\right]\right)+\frac{d^2}{dx^2}\left(\frac{\partial}{\partial y''}\left[ay''-y'^2+cxy\right]\right) &= 0\\
cx-\frac{d}{dx}\left(-2y'\right)+\frac{d^2}{dx^2}\left(a\right) &= 0\\
cx+2y''+0 &= 0\\
2y''+cx &= 0
\end{aligned}$$

(**d**) Using Eq. 12 (with x, y, y', y'' corresponding to s, ϕ, ϕ', ϕ'') we have:

$$\begin{aligned}
\frac{\partial}{\partial \phi}\left[\phi''^2+a\phi'-bs^2\phi+cs^3\right]-\frac{d}{ds}\left(\frac{\partial}{\partial \phi'}\left[\phi''^2+a\phi'-bs^2\phi+cs^3\right]\right)&\\
+\frac{d^2}{ds^2}\left(\frac{\partial}{\partial \phi''}\left[\phi''^2+a\phi'-bs^2\phi+cs^3\right]\right) &= 0\\
-bs^2-\frac{d}{ds}\left(a\right)+\frac{d^2}{ds^2}\left(2\phi''\right) &= 0\\
-bs^2-0+2\phi^{(4)} &= 0\\
2\phi^{(4)}-bs^2 &= 0
\end{aligned}$$

2. Obtain the Euler-Lagrange equations for the following functional integrals and solve them:
(**a**) $I\left[y\right]=\int_{x_1}^{x_2} y\left(2-y''\right)dx$.
(**b**) $I\left[y\right]=\int_{x_1}^{x_2}\left(xy+y'^2-x^2y''\right)dx$.
(**c**) $I\left[\theta\right]=\int_{t_1}^{t_2}\left(2\ddot{\theta}+\dot{\theta}^2-\omega^2\theta^2\right)dt$ (with ω being constant).
(**d**) $I\left[y\right]=\int_{x_1}^{x_2}\left(y''+\alpha y'^2-\beta\frac{y}{x}\right)dx$ (with α and β being constants).
Answer:
(**a**) In this case we have $F\left(x,y,y',y''\right)=y\left(2-y''\right)$ and hence the Euler-Lagrange equation (i.e. Eq. 12) is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y\left(2-y''\right)\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y\left(2-y''\right)\right]\right)+\frac{d^2}{dx^2}\left(\frac{\partial}{\partial y''}\left[y\left(2-y''\right)\right]\right) &= 0\\
\left(2-y''\right)-\frac{d}{dx}\left(0\right)+\frac{d^2}{dx^2}\left(-y\right) &= 0\\
2-y''-y'' &= 0\\
1-y'' &= 0\\
\frac{d^2y}{dx^2} &= 1
\end{aligned}$$

On integrating the last equation twice we obtain the solution, that is:

$$\begin{aligned}
\frac{dy}{dx} &= x+C\\
y &= \frac{1}{2}x^2+Cx+D
\end{aligned}$$

(**b**) In this case we have $F\left(x,y,y',y''\right)=xy+y'^2-x^2y''$ and hence the Euler-Lagrange equation (i.e. Eq. 12) is:

$$\frac{\partial}{\partial y}\left[xy+y'^2-x^2y''\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[xy+y'^2-x^2y''\right]\right)+\frac{d^2}{dx^2}\left(\frac{\partial}{\partial y''}\left[xy+y'^2-x^2y''\right]\right) = 0$$

$$\begin{aligned}
x - \frac{d}{dx}\left(2y'\right) + \frac{d^2}{dx^2}\left(-x^2\right) &= 0 \\
x - 2y'' - 2 &= 0 \\
\frac{d^2y}{dx^2} &= \frac{1}{2}x - 1
\end{aligned}$$

On integrating the last equation twice we obtain the solution, that is:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{4}x^2 - x + C \\
y &= \frac{1}{12}x^3 - \frac{1}{2}x^2 + Cx + D
\end{aligned}$$

(**c**) In this case we have $F(t,\theta,\dot{\theta},\ddot{\theta}) = 2\ddot{\theta} + \dot{\theta}^2 - \omega^2\theta^2$ and hence the Euler-Lagrange equation (i.e. Eq. 12 noting that x, y, y', y'' correspond to $t, \theta, \dot{\theta}, \ddot{\theta}$) is:

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left[2\ddot{\theta} + \dot{\theta}^2 - \omega^2\theta^2\right] - \frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}\left[2\ddot{\theta} + \dot{\theta}^2 - \omega^2\theta^2\right]\right) + \frac{d^2}{dt^2}\left(\frac{\partial}{\partial\ddot{\theta}}\left[2\ddot{\theta} + \dot{\theta}^2 - \omega^2\theta^2\right]\right) &= 0 \\
-2\omega^2\theta - \frac{d}{dt}\left(2\dot{\theta}\right) + \frac{d^2}{dt^2}(2) &= 0 \\
-2\omega^2\theta - 2\ddot{\theta} + 0 &= 0 \\
\ddot{\theta} + \omega^2\theta &= 0
\end{aligned}$$

So, the solution is $\theta = C\cos(\omega t) + D\sin(\omega t)$ (with C and D being constants) as can be checked by substitution into the Euler-Lagrange equation, that is:

$$\begin{aligned}
\frac{d^2}{dt^2}\big[C\cos(\omega t) + D\sin(\omega t)\big] + \omega^2\big[C\cos(\omega t) + D\sin(\omega t)\big] &\overset{?}{=} 0 \\
\frac{d}{dt}\big[-\omega C\sin(\omega t) + \omega D\cos(\omega t)\big] + \omega^2\big[C\cos(\omega t) + D\sin(\omega t)\big] &\overset{?}{=} 0 \\
\left[-\omega^2 C\cos(\omega t) - \omega^2 D\sin(\omega t)\right] + \omega^2\left[C\cos(\omega t) + D\sin(\omega t)\right] &\overset{?}{=} 0 \\
0 &= 0
\end{aligned}$$

(**d**) In this case we have $F\left(x,y,y',y''\right) = y'' + \alpha y'^2 - \beta\frac{y}{x}$ and hence the Euler-Lagrange equation (i.e. Eq. 12) is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y'' + \alpha y'^2 - \beta\frac{y}{x}\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'' + \alpha y'^2 - \beta\frac{y}{x}\right]\right) + \frac{d^2}{dx^2}\left(\frac{\partial}{\partial y''}\left[y'' + \alpha y'^2 - \beta\frac{y}{x}\right]\right) &= 0 \\
-\frac{\beta}{x} - \frac{d}{dx}\left(2\alpha y'\right) + \frac{d^2}{dx^2}(1) &= 0 \\
-\frac{\beta}{x} - 2\alpha y'' + 0 &= 0 \\
y'' + \frac{\beta}{2\alpha x} &= 0
\end{aligned}$$

So, the solution is $y = -\frac{\beta}{2\alpha}(x\ln x - x) + Cx + D$ (with C and D being constants) as can be checked by substitution into the Euler-Lagrange equation, that is:

$$\begin{aligned}
\left[-\frac{\beta}{2\alpha}(x\ln x - x) + Cx + D\right]'' + \frac{\beta}{2\alpha x} &\overset{?}{=} 0 \\
\left[-\frac{\beta}{2\alpha}\left(\ln x + \frac{x}{x} - 1\right) + C + 0\right]' + \frac{\beta}{2\alpha x} &\overset{?}{=} 0
\end{aligned}$$

$$\begin{aligned}\left[-\frac{\beta}{2\alpha}\left(\frac{1}{x}+0-0\right)+0\right]+\frac{\beta}{2\alpha x} &\overset{?}{=} 0\\ -\frac{\beta}{2\alpha x}+\frac{\beta}{2\alpha x} &= 0\end{aligned}$$

3. Find the extremizing (or stationarizing) functions of the following functional integrals as well as the specific solutions for the given boundary conditions:
   (**a**) $I[y]=\int_{\pi}^{\frac{3\pi}{2}}\left(x^2y''+y'^2-y^2\right)\,dx$ with $y(\pi)=2$ and $y\left(\frac{3\pi}{2}\right)=5$.
   (**b**) $I[y]=\int_0^1\left(y''^2+720x^2y\right)\,dx$ with $y(0)=0$, $y(1)=0$, $y'(0)=1$ and $y'(1)=1$.
   Answer:
   (**a**) We have $F=x^2y''+y'^2-y^2$ and hence from the Euler-Lagrange equation (Eq. 12) we get:

$$\begin{aligned}\frac{\partial}{\partial y}\left[x^2y''+y'^2-y^2\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[x^2y''+y'^2-y^2\right]\right)+\frac{d^2}{dx^2}\left(\frac{\partial}{\partial y''}\left[x^2y''+y'^2-y^2\right]\right) &= 0\\ -2y-\frac{d}{dx}(2y')+\frac{d^2}{dx^2}\left(x^2\right) &= 0\\ -2y-2y''+2 &= 0\\ y''+y-1 &= 0\end{aligned}$$

Hence, the extremizing function (which can be verified by substitution in the last equation) is:

$$y=C\cos x+D\sin x+1$$

From the given boundary conditions (respectively), we get:

$$\begin{aligned}C\cos\pi+D\sin\pi+1 &= 2 &\quad\to\quad& C=-1\\ C\cos\frac{3\pi}{2}+D\sin\frac{3\pi}{2}+1 &= 5 &\quad\to\quad& D=-4\end{aligned}$$

Therefore, the specific solution is:

$$y=-\cos x-4\sin x+1$$

(**b**) We have $F=y''^2+720x^2y$ and hence from the Euler-Lagrange equation (Eq. 12) we get:

$$\begin{aligned}\frac{\partial}{\partial y}\left[y''^2+720x^2y\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y''^2+720x^2y\right]\right)+\frac{d^2}{dx^2}\left(\frac{\partial}{\partial y''}\left[y''^2+720x^2y\right]\right) &= 0\\ 720x^2-\frac{d}{dx}(0)+\frac{d^2}{dx^2}(2y'') &= 0\\ 720x^2+2y^{(4)} &= 0\\ y^{(4)} &= -360x^2\end{aligned}$$

On integrating 4 times we get the extremizing function:

$$y=-x^6+C_3x^3+C_2x^2+C_1x+C_0$$

From the given boundary conditions (respectively), we get:

$$\begin{aligned}-0+0+0+0+C_0 &= 0 &\quad\to\quad& C_0=0\\ -1+C_3+C_2+C_1+0 &= 0 &\quad\to\quad& C_3+C_2+C_1=1\\ -0+0+0+C_1 &= 1 &\quad\to\quad& C_1=1\\ -6+3C_3+2C_2+1 &= 1 &\quad\to\quad& 3C_3+2C_2=6\end{aligned}$$

Accordingly: $C_0=0$, $C_1=1$, $C_2=-6$ and $C_3=6$. Therefore, the specific solution is:

$$y=-x^6+6x^3-6x^2+x$$

1.6 Variational Problems with Multiple Independent Variables

In some variational problems (i.e. multi-dimensional problems) the extremizing function y depends on more than one independent variable (unlike the previous problems where y depends solely on x). In such cases, the Euler-Lagrange equation is modified to account for the additional dependency. For example, if $y \equiv y(x_1, x_2)$ and hence:[22]

$$I\left[y\right] = \iint_{\Omega} F\left(x_1, x_2, y, y_{x_1}, y_{x_2}\right) \, dx_1 \, dx_2 \tag{14}$$

(where $y_{x_1} = \partial y/\partial x_1$ and $y_{x_2} = \partial y/\partial x_2$ and where y is identified on the boundary $\partial\Omega$ of the domain Ω of the integral) then the Euler-Lagrange equation[23] becomes:

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial y_{x_1}}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial y_{x_2}}\right) = 0 \tag{15}$$

It is very important to note that the notation in the formulation of variational problems with multiple independent variables is rather misleading. The reason is that: based on the derivation of Eq. 15 (and its generalizations) the partial derivatives with respect to the independent variables (i.e. $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$) apply to all the dependencies of their operands (i.e. $\frac{\partial F}{\partial y_{x_1}}$ and $\frac{\partial F}{\partial y_{x_2}}$) on the independent variables (i.e. x_1 and x_2) whether these dependencies are explicit or implicit, and hence from this perspective they are like total derivatives. However, they are not notated as total derivatives because the expressions in these formulations have other dependencies and hence $\frac{d}{dx_1}$ is not appropriate due to the existence of x_2 dependency (and similarly $\frac{d}{dx_2}$ is not appropriate due to the existence of x_1 dependency). In fact, some texts distinguish these partial derivatives by the notation $\frac{D}{Dx_1}$ and $\frac{D}{Dx_2}$ and hence Eq. 15 is written as:

$$\frac{\partial F}{\partial y} - \frac{D}{Dx_1}\left(\frac{\partial F}{\partial y_{x_1}}\right) - \frac{D}{Dx_2}\left(\frac{\partial F}{\partial y_{x_2}}\right) = 0 \tag{16}$$

The effect of this difference in the meaning of the notation will be seen in the solution of some Problems of this section. For example, in the solution of part (a) of Problem 2 we did not set $\frac{\partial}{\partial x}\left(y_x\right)$ and $\frac{\partial}{\partial z}\left(y_z\right)$ to zero (as we usually do in the formulations of the Euler-Lagrange equation where we consider only explicit dependencies), but we did (for the purpose of clarification) as follows:

$$\begin{aligned} \frac{\partial}{\partial x}\left(y_x\right) - \frac{\partial}{\partial z}\left(y_z\right) &= 0 \\ \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) - \frac{\partial}{\partial z}\left(\frac{\partial y}{\partial z}\right) &= 0 \\ \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial z^2} &= 0 \\ y_{xx} - y_{zz} &= 0 \end{aligned}$$

As we see, the derivative $\frac{\partial}{\partial x}$ acts even on the implicit x dependency of y_x (and similarly for $\frac{\partial}{\partial z}$ which acts even on the implicit z dependency of y_z).

Problems

1. Write down the functional integral I and the Euler-Lagrange equation when the extremizing function y depends on n independent variables $x_1, \cdots, x_n$.
   Answer:

$$I\left[y\right] = \int \!\! \cdots \!\! \int_{\Omega} F\left(x_1, \cdots, x_n, y, y_{x_1}, \cdots, y_{x_n}\right) \, dx_1 \cdots dx_n$$

---

[22] The reader should note that the functional in this case takes the form of a double integral on a given 2D domain Ω.

[23] Although we call this equation (and its alike) the Euler-Lagrange equation it is not really the Euler-Lagrange equation but an extension (or generalization) of it. In fact, in some texts this equation (and its alike) is labeled differently (e.g. Euler-Ostrogradsky) but we keep our labeling (i.e. Euler-Lagrange equation) for simplicity and to avoid possible confusion.

$$\frac{\partial F}{\partial y} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_{x_i}}\right) = 0$$

where $y_{x_i} = \partial y / \partial x_i$.

2. Find the Euler-Lagrange equations for the following functional integrals:
(**a**) $I[y] = \iint_\Omega \left(y_x^2 - y_z^2\right)\, dx\, dz$.
(**b**) $I[y] = \iint_\Omega \left(y_t^2 + y_x^2 + Cy\right)\, dt\, dx$.
(**c**) $I[y] = \iint_\Omega \left(x^2 y_x^2 - a z^2 y_z^2\right)\, dx\, dz$ (with a being a constant).
(**d**) $I[\xi] = \iint_\Omega \left(\xi_x^2 + \xi_y^2 + \alpha\xi\right)\, dx\, dy$ $\left[\text{with } \alpha \text{ being a function of } x \text{ and } y\text{, i.e. } \alpha = \alpha(x,y)\right]$.
Answer:
(**a**) Using Eq. 15 with $F(x, z, y, y_x, y_z) = y_x^2 - y_z^2$ (noting that $x_1, x_2, y, y_{x_1}, y_{x_2}$ in Eq. 15 correspond to x, z, y, y_x, y_z here) we get:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y_x^2 - y_z^2\right] - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y_x}\left[y_x^2 - y_z^2\right]\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y_z}\left[y_x^2 - y_z^2\right]\right) &= 0 \\
0 - \frac{\partial}{\partial x}(2y_x) - \frac{\partial}{\partial z}(-2y_z) &= 0 \\
\frac{\partial}{\partial x}(y_x) - \frac{\partial}{\partial z}(y_z) &= 0 \\
\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) - \frac{\partial}{\partial z}\left(\frac{\partial y}{\partial z}\right) &= 0 \\
\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial z^2} &= 0 \\
y_{xx} - y_{zz} &= 0
\end{aligned}$$

(**b**) Using Eq. 15 with $F(t, x, y, y_t, y_x) = y_t^2 + y_x^2 + Cy$ (noting that $x_1, x_2, y, y_{x_1}, y_{x_2}$ in Eq. 15 correspond to t, x, y, y_t, y_x here) we get:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y_t^2 + y_x^2 + Cy\right] - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial y_t}\left[y_t^2 + y_x^2 + Cy\right]\right) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y_x}\left[y_t^2 + y_x^2 + Cy\right]\right) &= 0 \\
C - \frac{\partial}{\partial t}(2y_t) - \frac{\partial}{\partial x}(2y_x) &= 0 \\
\frac{\partial}{\partial t}(y_t) + \frac{\partial}{\partial x}(y_x) &= \frac{C}{2} \\
\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) &= \frac{C}{2} \\
\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} &= \frac{C}{2}
\end{aligned}$$

which is a 2D Poisson equation.
(**c**) Using Eq. 15 with $F(x, z, y, y_x, y_z) = x^2 y_x^2 - a z^2 y_z^2$ (noting that $x_1, x_2, y, y_{x_1}, y_{x_2}$ in Eq. 15 correspond to x, z, y, y_x, y_z here) we get:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[x^2 y_x^2 - a z^2 y_z^2\right] - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y_x}\left[x^2 y_x^2 - a z^2 y_z^2\right]\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y_z}\left[x^2 y_x^2 - a z^2 y_z^2\right]\right) &= 0 \\
0 - \frac{\partial}{\partial x}\left(2x^2 y_x\right) - \frac{\partial}{\partial z}\left(-2a z^2 y_z\right) &= 0 \\
-2\left(2x y_x + x^2 y_{xx}\right) + 2a\left(2z y_z + z^2 y_{zz}\right) &= 0 \\
-2x y_x - x^2 y_{xx} + 2a z y_z + a z^2 y_{zz} &= 0
\end{aligned}$$

(**d**) Using Eq. 15 with $F(x, y, \xi, \xi_x, \xi_y) = \xi_x^2 + \xi_y^2 + \alpha\xi$ (noting that $x_1, x_2, y, y_{x_1}, y_{x_2}$ in Eq. 15

correspond to x, y, ξ, ξ_x, ξ_y here) we get:

$$\begin{aligned}
\frac{\partial}{\partial \xi}\left[\xi_x^2+\xi_y^2+\alpha\xi\right]-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial \xi_x}\left[\xi_x^2+\xi_y^2+\alpha\xi\right]\right)-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial \xi_y}\left[\xi_x^2+\xi_y^2+\alpha\xi\right]\right) &= 0\\
\alpha-\frac{\partial}{\partial x}\left(2\xi_x\right)-\frac{\partial}{\partial y}\left(2\xi_y\right) &= 0\\
\alpha-2\xi_{xx}-2\xi_{yy} &= 0\\
\xi_{xx}+\xi_{yy} &= \frac{\alpha}{2}\\
\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2} &= \frac{\alpha}{2}
\end{aligned}$$

which is a 2D Poisson equation.

3. Find the Euler-Lagrange equation for the functional $I[y]=\iint_\Omega\left(y_x^2+y_z^2\right)dx\,dz$ and suggest a solution.
Answer: Using Eq. 15 with $F(x,z,y,y_x,y_z)=y_x^2+y_z^2$ (noting that $x_1,x_2,y,y_{x_1},y_{x_2}$ in Eq. 15 correspond to x,z,y,y_x,y_z here) we get:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y_x^2+y_z^2\right]-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y_x}\left[y_x^2+y_z^2\right]\right)-\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y_z}\left[y_x^2+y_z^2\right]\right) &= 0\\
0-\frac{\partial}{\partial x}\left(2y_x\right)-\frac{\partial}{\partial z}\left(2y_z\right) &= 0\\
\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial y}{\partial z}\right) &= 0\\
\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial z^2} &= 0
\end{aligned}$$

which is a 2D Laplace equation. One possible solution is $y=xz$ because:

$$\frac{\partial^2}{\partial x^2}(xz)+\frac{\partial^2}{\partial z^2}(xz)=\frac{\partial}{\partial x}(z)+\frac{\partial}{\partial z}(x)=0+0=0$$

Note: it should be obvious that the independent variables x and z are independent of each other.

4. The area of a simple surface in 3D Euclidean space with a given domain Ω and a given closed boundary curve is to be optimized.[24] Find the Euler-Lagrange equation for this problem.
Answer: Let the surface be given as $z=z(x,y)$ over the domain Ω (in the xy plane) with a given closed boundary space curve Γ (where the projection of Γ on the xy plane is a simple plane curve $\partial\Omega$). So, the problem is a variational problem with two independent variables (i.e. x and y) and one dependent variable (i.e. z). Now, from elementary calculus we know that the area of such a surface is given by:

$$\sigma=\iint_\Omega\sqrt{1+z_x^2+z_y^2}\,dx\,dy$$

where $z_x=\partial z/\partial x$ and $z_y=\partial z/\partial y$. So, our functional integral is $I[z]\equiv\sigma$ and hence $F(x,y,z,z_x,z_y)=\sqrt{1+z_x^2+z_y^2}$. Accordingly, the Euler-Lagrange equation for this Problem is (see Eq. 15 noting that x,y,z,z_x,z_y here correspond to $x_1,x_2,y,y_{x_1},y_{x_2}$ in Eq. 15):

$$\begin{aligned}
\frac{\partial}{\partial z}\sqrt{1+z_x^2+z_y^2}-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z_x}\sqrt{1+z_x^2+z_y^2}\right)-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z_y}\sqrt{1+z_x^2+z_y^2}\right) &= 0\\
0-\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)-\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) &= 0
\end{aligned}$$

---

[24] "Optimized" here should mean minimized since the area of such a surface can diverge.

$$\begin{aligned}
\left(\frac{z_{xx}}{\sqrt{1+z_x^2+z_y^2}} - \frac{z_x\left(2z_xz_{xx}+2z_yz_{yx}\right)}{2\left(1+z_x^2+z_y^2\right)^{3/2}}\right) + \left(\frac{z_{yy}}{\sqrt{1+z_x^2+z_y^2}} - \frac{z_y\left(2z_xz_{xy}+2z_yz_{yy}\right)}{2\left(1+z_x^2+z_y^2\right)^{3/2}}\right) &= 0\\
\left(\frac{z_{xx}\left(1+z_x^2+z_y^2\right)}{\left(1+z_x^2+z_y^2\right)^{3/2}} - \frac{z_x\left(z_xz_{xx}+z_yz_{yx}\right)}{\left(1+z_x^2+z_y^2\right)^{3/2}}\right) + \left(\frac{z_{yy}\left(1+z_x^2+z_y^2\right)}{\left(1+z_x^2+z_y^2\right)^{3/2}} - \frac{z_y\left(z_xz_{xy}+z_yz_{yy}\right)}{\left(1+z_x^2+z_y^2\right)^{3/2}}\right) &= 0\\
\left(\frac{z_{xx}+z_{xx}z_x^2+z_{xx}z_y^2-z_x^2z_{xx}-z_xz_yz_{yx}}{\left(1+z_x^2+z_y^2\right)^{3/2}}\right) + \left(\frac{z_{yy}+z_{yy}z_x^2+z_{yy}z_y^2-z_xz_yz_{xy}-z_y^2z_{yy}}{\left(1+z_x^2+z_y^2\right)^{3/2}}\right) &= 0\\
\frac{z_{xx}+z_{xx}z_y^2-z_xz_yz_{yx}}{\left(1+z_x^2+z_y^2\right)^{3/2}} + \frac{z_{yy}+z_{yy}z_x^2-z_xz_yz_{xy}}{\left(1+z_x^2+z_y^2\right)^{3/2}} &= 0\\
\frac{z_{xx}+z_{xx}z_y^2-2z_xz_yz_{xy}+z_{yy}+z_{yy}z_x^2}{\left(1+z_x^2+z_y^2\right)^{3/2}} &= 0\\
z_{xx}+z_{xx}z_y^2-2z_xz_yz_{xy}+z_{yy}+z_{yy}z_x^2 &= 0 \quad (17)
\end{aligned}$$

We remark that in line 3 the partial differentiation with respect to x and y includes implicit as well as explicit dependencies on these variables (as explained in the text), while in line 7 we used $z_{yx} = z_{xy}$.
Note: Eq. 17 defines the provision for minimal surfaces[25] with the above given conditions (noting that some minimal surfaces may require a slight modification to the above conditions with regard to the boundary).

5. Show that planes are minimal surfaces.
   Answer: A plane in 3D Euclidean space (defined over a given domain in the xy plane with a given boundary) is defined by the equation $z = ax + by + c$ (with a, b, c being constants).[26] Accordingly, $z_{xx} = z_{xy} = z_{yy} = 0$ and hence Eq. 17 is satisfied identically. So, the plane is a solution to Eq. 17 and hence its area is minimum (according to Problem 4), i.e. it is a minimal surface, as required.
6. Find the extremizing (or stationarizing) function of the following functional integral and suggest a specific solution that satisfies the given boundary conditions (as well as the given constraint):

$$I[z] = \int_0^1 \int_0^1 \left(z_x^2 - z_y^2\right) dx\,dy \qquad z(0,y) = z(x,0) = z(1,y) = z(x,1) = 0 \qquad z(0.5, 0.5) = 1$$

Answer:[27] Using Eq. 15 with $F(x, y, z, z_x, z_y) = z_x^2 - z_y^2$ (noting that $x_1, x_2, y, y_{x_1}, y_{x_2}$ in Eq. 15 correspond to x, y, z, z_x, z_y here) we get:

$$\begin{aligned}
\frac{\partial}{\partial z}\left[z_x^2 - z_y^2\right] - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z_x}\left[z_x^2 - z_y^2\right]\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z_y}\left[z_x^2 - z_y^2\right]\right) &= 0\\
0 - \frac{\partial}{\partial x}\left(2z_x\right) - \frac{\partial}{\partial y}\left(-2z_y\right) &= 0\\
z_{xx} - z_{yy} &= 0
\end{aligned}$$

This is the Euler-Lagrange equation of this Problem. We can suggest the following solution (which satisfies all the given boundary conditions and constraint as well as the Euler-Lagrange equation):

$$z = \sin(\pi x)\sin(\pi y)$$

Note: this solution is plotted later in the book (see the lower frame of Figure 73).

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[25] Minimal surface is a surface whose area is minimum compared to the area of any other surface that shares the same boundary curve. Common examples of minimal surface are planes, catenoids, helicoids and ennepers.

[26] The fact that some planes may be defined differently (e.g. $y = 6$) does not affect the generality of our assertion because with a simple transformation (which does not affect the geometrical properties) of the plane or the coordinate system (e.g. rotation) the plane can be defined by an equation of the above form.

[27] In this answer (as well as in the answers of the following Problems in this section), we omit many details and possibilities to avoid exceeding our space and purpose.

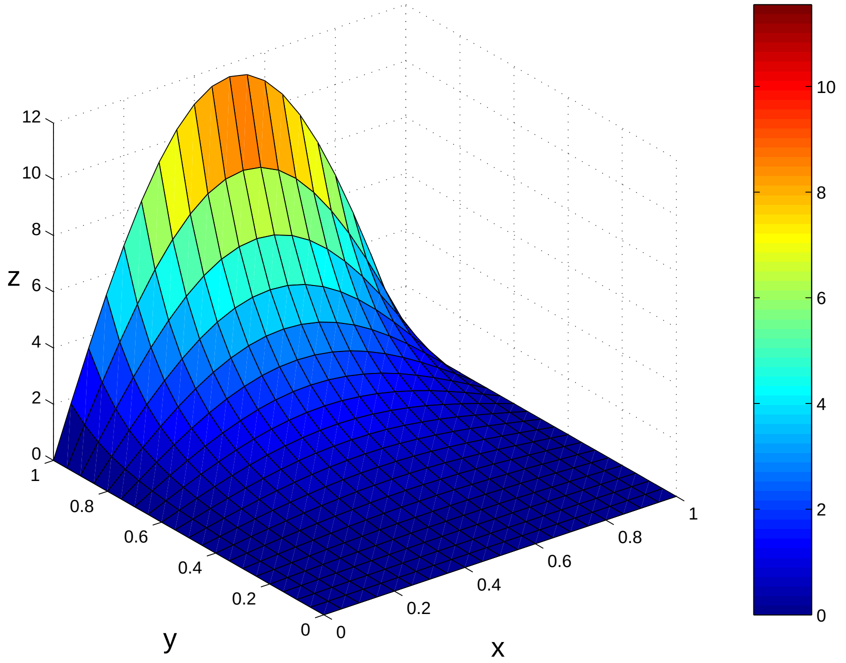

Figure 2: The solution $z = \sin(\pi x)\sinh(\pi y)$ of Problem 7 of § 1.6.

7. Find the extremizing (or stationarizing) function of the following functional integral and suggest a specific solution that satisfies the given boundary conditions and plot the solution:

$$I[z] = \int_0^1 \int_0^1 \left(z_x^2 + z_y^2\right) dx\,dy \qquad z(0,y) = z(x,0) = z(1,y) = 0 \qquad z(x,1) = \sin(\pi x)\sinh(\pi)$$

Answer: Using Eq. 15 with $F(x,y,z,z_x,z_y) = z_x^2 + z_y^2$ (noting that $x_1, x_2, y, y_{x_1}, y_{x_2}$ in Eq. 15 correspond to x, y, z, z_x, z_y here) we get:

$$\begin{aligned} \frac{\partial}{\partial z}\left[z_x^2 + z_y^2\right] - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z_x}\left[z_x^2 + z_y^2\right]\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z_y}\left[z_x^2 + z_y^2\right]\right) &= 0 \\ 0 - \frac{\partial}{\partial x}(2z_x) - \frac{\partial}{\partial y}(2z_y) &= 0 \\ z_{xx} + z_{yy} &= 0 \end{aligned}$$

This is the Euler-Lagrange equation of this Problem. We can suggest the following solution (which satisfies all the given boundary conditions as well as the Euler-Lagrange equation):

$$z = \sin(\pi x)\sinh(\pi y)$$

This solution is plotted in Figure 2.

8. Find the extremizing (or stationarizing) function of the following functional integral and suggest a specific solution that satisfies the given boundary conditions:

$$I[z] = \int_0^1 \int_0^1 \left(z_x^2 + z_y^2 - \pi^2 z^2 + \frac{2z^2}{y^2}\right) dx\,dy \qquad z(0,y) = z(x,0) = z(1,y) = 0 \qquad z(x,1) = \sin(\pi x)$$

Answer: Using Eq. 15 with $F(x,y,z,z_x,z_y) = z_x^2 + z_y^2 - \pi^2 z^2 + \frac{2z^2}{y^2}$ (noting that $x_1, x_2, y, y_{x_1}, y_{x_2}$ in Eq. 15 correspond to x, y, z, z_x, z_y here) we get:

$$\frac{\partial}{\partial z}\left[z_x^2 + z_y^2 - \pi^2 z^2 + \frac{2z^2}{y^2}\right] - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z_x}\left[z_x^2 + z_y^2 - \pi^2 z^2 + \frac{2z^2}{y^2}\right]\right)$$

$$\begin{aligned} -\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z_y}\left[z_x^2+z_y^2-\pi^2 z^2+\frac{2z^2}{y^2}\right]\right) &= 0 \\ -2\pi^2 z+\frac{4z}{y^2}-\frac{\partial}{\partial x}\left(2z_x\right)-\frac{\partial}{\partial y}\left(2z_y\right) &= 0 \\ -2\pi^2 z+\frac{4z}{y^2}-2z_{xx}-2z_{yy} &= 0 \\ z_{xx}+z_{yy}+\pi^2 z-\frac{2z}{y^2} &= 0 \end{aligned}$$

This is the Euler-Lagrange equation of this Problem. We can suggest the following solution (which satisfies all the given boundary conditions as well as the Euler-Lagrange equation):

$$z = y^2 \sin\left(\pi x\right)$$

Note: this solution is plotted later in the book (see the lower frame of Figure 74).

9. Find the extremizing (or stationarizing) function of the following functional integral and suggest a specific solution that satisfies the given boundary conditions and plot the solution:

$$I\left[z\right] = \int_0^1 \int_0^1 \left(z_x^2+z_y^2-4z\right)\, dx\, dy$$

$$z\left(0,y\right)=0 \qquad z\left(x,0\right)=-x^2 \qquad z\left(1,y\right)=y-1 \qquad z\left(x,1\right)=x-x^2$$

Answer: Using Eq. 15 with $F(x,y,z,z_x,z_y) = z_x^2+z_y^2-4z$ (noting that $x_1, x_2, y, y_{x_1}, y_{x_2}$ in Eq. 15 correspond to x, y, z, z_x, z_y here) we get:

$$\begin{aligned} \frac{\partial}{\partial z}\left[z_x^2+z_y^2-4z\right]-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z_x}\left[z_x^2+z_y^2-4z\right]\right)-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z_y}\left[z_x^2+z_y^2-4z\right]\right) &= 0 \\ -4-\frac{\partial}{\partial x}\left(2z_x\right)-\frac{\partial}{\partial y}\left(2z_y\right) &= 0 \\ z_{xx}+z_{yy}+2 &= 0 \end{aligned}$$

This is the Euler-Lagrange equation of this Problem. We can suggest the following solution (which satisfies all the given boundary conditions as well as the Euler-Lagrange equation):

$$z = xy - x^2$$

This solution is plotted in Figure 3.

10. Find the extremizing (or stationarizing) function of the following functional integral and suggest a specific solution that satisfies the given boundary conditions:

$$I\left[z\right] = \int_0^1 \int_0^1 \left(z_x^2+z_y^2-4z\right)\, dx\, dy \qquad\qquad z\left(0,y\right)=z\left(x,0\right)=z\left(1,y\right)=z\left(x,1\right)=0$$

Answer: The functional integral of this Problem is the same as the functional integral of Problem 9 and hence the Euler-Lagrange equation for this Problem is the same as for Problem 9. We can suggest the following series solution:

$$z = \frac{8}{\pi^4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left(1-\left[-1\right]^m\right)\left(1-\left[-1\right]^n\right)}{mn\left(m^2+n^2\right)}\sin\left(m\pi x\right)\sin\left(n\pi y\right)$$

Note: this solution is plotted (up to and including $m = n = 7$) later in the book (see the lower frame of Figure 75).

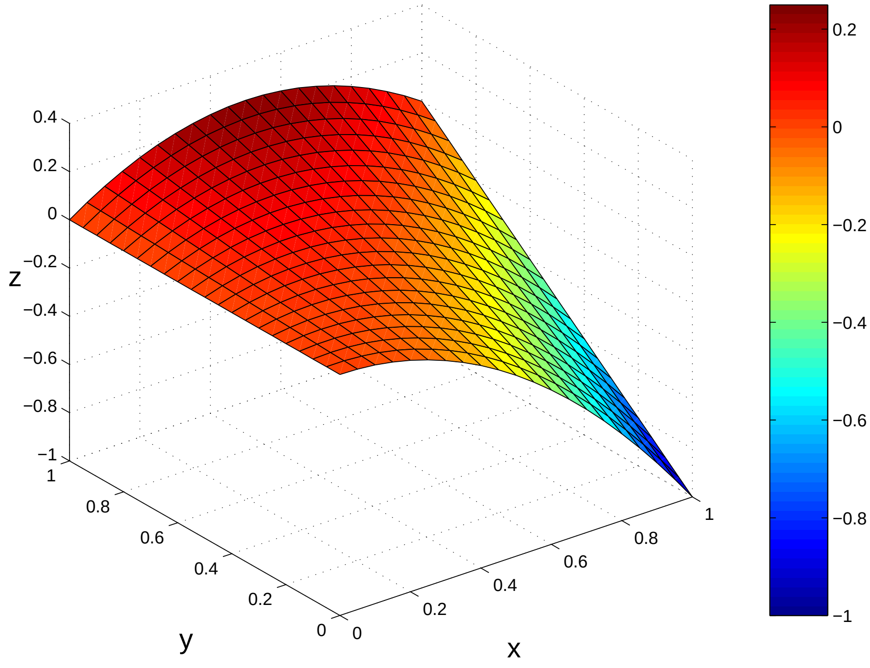

Figure 3: The solution $z = xy - x^2$ of Problem 9 of § 1.6.

1.7 Variational Problems with Multiple Dependent Variables

In some variational problems the functional I depends on more than one dependent variable (i.e. extremizing function). For example, I may depend on two extremizing functions $y_1 \equiv y_1(x)$ and $y_2 \equiv y_2(x)$ and hence I (which is originally given by Eq. 1) becomes:

$$I[y_1, y_2] = \int_{x_1}^{x_2} F(x, y_1, y_2, y_1', y_2')\, dx \tag{18}$$

where I is the functional integral whose optimization depends on the extremizing functions y_1 and y_2, the prime stands for d/dx, and the functional is constrained by four boundary conditions: $y_1(x_1) = C_1$, $y_1(x_2) = C_2$, $y_2(x_1) = D_1$, $y_2(x_2) = D_2$ (with C_1, C_2, D_1, D_2 being constants). In this case there will be one Euler-Lagrange equation (see Eq. 2) for each extremizing function, that is:

$$\frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right) = 0 \tag{19}$$

$$\frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right) = 0 \tag{20}$$

where these equations should be solved simultaneously[28] to obtain the solution of the variational problem. The above formulation can be easily generalized when the functional depends on more than two extremizing functions (see Problem 1; also see § 6).

Problems

1. Write down the functional integral I and the Euler-Lagrange equations when I depends on n dependent variables $y_1(x), \cdots, y_n(x)$.
Answer:

$$I[y_1, \cdots, y_n] = \int_{x_1}^{x_2} F(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\, dx$$

---

[28] Simultaneous here does not necessarily mean they are linked as a system.

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0 \qquad (i = 1, \cdots, n)$$

2. Obtain the Euler-Lagrange equations for the following functional integrals (of multiple dependent variables):
(**a**) $I[y,z] = \int_{x_1}^{x_2} \left(y'^3 + az'^2 + 3bxy - cz\right) dx$ (with a, b, c being constants).
(**b**) $I[r,\phi] = \int_{s_1}^{s_2} \left(r'^2 + Cr^{-1} + r^2\phi'^2\right) ds$.
(**c**) $I[x_1,x_2] = \int_{t_1}^{t_2} \left(\dot{x}_1^2 + \dot{x}_2^2\right) dt$.
Answer:
(**a**) We have $F(x,y,z,y',z') = y'^3 + az'^2 + 3bxy - cz$ and hence the Euler-Lagrange equation for y is:

$$\begin{aligned} \frac{\partial}{\partial y}\left[y'^3 + az'^2 + 3bxy - cz\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^3 + az'^2 + 3bxy - cz\right]\right) &= 0 \\ 3bx - \frac{d}{dx}\left(3y'^2\right) &= 0 \\ 3bx - 6y'y'' &= 0 \\ 6y'y'' - 3bx &= 0 \end{aligned}$$

while the Euler-Lagrange equation for z is:

$$\begin{aligned} \frac{\partial}{\partial z}\left[y'^3 + az'^2 + 3bxy - cz\right] - \frac{d}{dx}\left(\frac{\partial}{\partial z'}\left[y'^3 + az'^2 + 3bxy - cz\right]\right) &= 0 \\ -c - \frac{d}{dx}\left(2az'\right) &= 0 \\ -c - 2az'' &= 0 \\ 2az'' + c &= 0 \end{aligned}$$

(**b**) We have $F(s,r,\phi,r',\phi') = r'^2 + Cr^{-1} + r^2\phi'^2$ and hence the Euler-Lagrange equation for r is:

$$\begin{aligned} \frac{\partial}{\partial r}\left[r'^2 + Cr^{-1} + r^2\phi'^2\right] - \frac{d}{ds}\left(\frac{\partial}{\partial r'}\left[r'^2 + Cr^{-1} + r^2\phi'^2\right]\right) &= 0 \\ -Cr^{-2} + 2r\phi'^2 - \frac{d}{ds}\left(2r'\right) &= 0 \\ -Cr^{-2} + 2r\phi'^2 - 2r'' &= 0 \\ 2r'' + Cr^{-2} - 2r\phi'^2 &= 0 \end{aligned}$$

while the Euler-Lagrange equation for ϕ is:

$$\begin{aligned} \frac{\partial}{\partial \phi}\left[r'^2 + Cr^{-1} + r^2\phi'^2\right] - \frac{d}{ds}\left(\frac{\partial}{\partial \phi'}\left[r'^2 + Cr^{-1} + r^2\phi'^2\right]\right) &= 0 \\ 0 - \frac{d}{ds}\left(2r^2\phi'\right) &= 0 \\ -\left(4rr'\phi' + 2r^2\phi''\right) &= 0 \\ 2rr'\phi' + r^2\phi'' &= 0 \end{aligned}$$

(**c**) We have $F(t,x_1,x_2,\dot{x}_1,\dot{x}_2) = \dot{x}_1^2 + \dot{x}_2^2$ and hence the Euler-Lagrange equation for x_1 is:

$$\begin{aligned} \frac{\partial}{\partial x_1}\left[\dot{x}_1^2 + \dot{x}_2^2\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_1}\left[\dot{x}_1^2 + \dot{x}_2^2\right]\right) &= 0 \\ 0 - \frac{d}{dt}\left(2\dot{x}_1\right) &= 0 \end{aligned}$$

$$\ddot{x}_1 = 0$$

Similarly, the Euler-Lagrange equation for x_2 is $\ddot{x}_2 = 0$. So, the Euler-Lagrange equations are:

$$\ddot{x}_i = 0 \qquad (i = 1, 2)$$

3. Find the extremizing (or stationarizing) functions of the following functional integrals as well as the specific solutions for the given boundary conditions:
(**a**) $I[y_1, y_2] = \int_0^1 \left(y_1'^2 + y_1^2 + y_2'^2 + 4y_2\right) dx$ with $y_1(0) = 1$, $y_1(1) = 0$, $y_2(0) = 1$, $y_2(1) = 0$.
(**b**) $I[y_1, y_2] = \int_0^{\pi/2} \left(y_1'^2 - y_1^2 + y_2'^2 + y_2^2\right) dx$ with $y_1(0) = 1$, $y_1(\pi/2) = 1$, $y_2(0) = 1$, $y_2(\pi/2) = 1$.
Answer:
(**a**) We have $F = y_1'^2 + y_1^2 + y_2'^2 + 4y_2$ and hence from the Euler-Lagrange equations (Eqs. 19 and 20) we get:

$$\begin{aligned}
\frac{\partial}{\partial y_1}\left[y_1'^2 + y_1^2 + y_2'^2 + 4y_2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y_1'}\left[y_1'^2 + y_1^2 + y_2'^2 + 4y_2\right]\right) &= 0\\
2y_1 - \frac{d}{dx}(2y_1') &= 0\\
2y_1 - 2y_1'' &= 0\\
y_1'' - y_1 &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial y_2}\left[y_1'^2 + y_1^2 + y_2'^2 + 4y_2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y_2'}\left[y_1'^2 + y_1^2 + y_2'^2 + 4y_2\right]\right) &= 0\\
4 - \frac{d}{dx}(2y_2') &= 0\\
4 - 2y_2'' &= 0\\
y_2'' &= 2
\end{aligned}$$

Hence, $y_1 = C_1 \cosh x + D_1 \sinh x$ and $y_2 = x^2 + C_2 x + D_2$.
From the given boundary conditions (respectively), we get:

$$\begin{aligned}
C_1 \cosh 0 + D_1 \sinh 0 &= 1 & &\rightarrow & C_1 &= 1\\
C_1 \cosh 1 + D_1 \sinh 1 &= 0 & &\rightarrow & D_1 &= -\coth 1\\
0 + 0 + D_2 &= 1 & &\rightarrow & D_2 &= 1\\
1 + C_2 + D_2 &= 0 & &\rightarrow & C_2 &= -2
\end{aligned}$$

Therefore, the specific solution is:

$$y_1 = \cosh x - (\coth 1)\sinh x \qquad\qquad y_2 = x^2 - 2x + 1$$

(**b**) We have $F = y_1'^2 - y_1^2 + y_2'^2 + y_2^2$ and hence from the Euler-Lagrange equations (Eqs. 19 and 20) we get:

$$\begin{aligned}
\frac{\partial}{\partial y_1}\left[y_1'^2 - y_1^2 + y_2'^2 + y_2^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y_1'}\left[y_1'^2 - y_1^2 + y_2'^2 + y_2^2\right]\right) &= 0\\
-2y_1 - \frac{d}{dx}(2y_1') &= 0\\
-2y_1 - 2y_1'' &= 0\\
y_1'' + y_1 &= 0
\end{aligned}$$

$$\frac{\partial}{\partial y_2}\left[y_1'^2 - y_1^2 + y_2'^2 + y_2^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y_2'}\left[y_1'^2 - y_1^2 + y_2'^2 + y_2^2\right]\right) = 0$$

$$\begin{aligned} 2y_2 - \frac{d}{dx}(2y_2') &= 0 \\ 2y_2 - 2y_2'' &= 0 \\ y_2'' - y_2 &= 0 \end{aligned}$$

Hence, $y_1 = C_1 \cos x + D_1 \sin x$ and $y_2 = C_2 \cosh x + D_2 \sinh x$.
From the given boundary conditions (respectively), we get:

$$\begin{aligned} C_1 \cos 0 + D_1 \sin 0 &= 1 && \rightarrow && C_1 = 1 \\ C_1 \cos\frac{\pi}{2} + D_1 \sin\frac{\pi}{2} &= 1 && \rightarrow && D_1 = 1 \\ C_2 \cosh 0 + D_2 \sinh 0 &= 1 && \rightarrow && C_2 = 1 \\ C_2 \cosh\frac{\pi}{2} + D_2 \sinh\frac{\pi}{2} &= 1 && \rightarrow && D_2 = \operatorname{csch}\frac{\pi}{2} - \coth\frac{\pi}{2} \end{aligned}$$

Therefore, the specific solution is:

$$y_1 = \cos x + \sin x \qquad\qquad y_2 = \cosh x + \left(\operatorname{csch}\frac{\pi}{2} - \coth\frac{\pi}{2}\right)\sinh x$$

1.8 Variational Problems with Constraints

In some variational problems certain constraints are imposed on the sought solution and this requires embedding these constraints as additional conditions within the variational formulation using a certain technique. However, before we talk about this type of variational problems in the calculus of variations we need to talk briefly about the Lagrange multipliers technique in general which is usually used in dealing with this type of variational problems in the calculus of variations.

The Lagrange multipliers technique is an analytical method used in the search for extremums of functions when certain constraints are imposed on these functions. The theoretical basis of this technique is that a function $f(x)$ on which a given constraint $g(x) = c$ (with c being a constant) is imposed has an extremum at a point P *iff* the function $f + \lambda g$ has an extremum at P (where the parameter λ is called the Lagrange multiplier).[29] So, instead of extremizing f in our search we extremize $h \equiv f + \lambda g$ (so that the constraint is included in the formulation). More than one constraint can be embedded using more than one Lagrange multiplier (i.e. one Lagrange multiplier for each constraint). For example, if a function f is to be extremized subject to three constraints g_1, g_2, g_3 (i.e. $g_1 = c_1$, $g_2 = c_2$, and $g_3 = c_3$) then the extremization formulation (according to the technique of Lagrange multipliers) will take the form $f + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3$ (with $\lambda_1, \lambda_2, \lambda_3$ being the Lagrange multipliers), and hence we should extremize $h \equiv f + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3$ (instead of f) in our search.

Now, returning to the issue of the variational problems with constraints in the calculus of variations it should be obvious that the Lagrange multipliers technique as described above (with some simple adaptations) can be used to deal with this sort of problems in the calculus of variations. Accordingly, we can say: a functional $I_1 = \int_{x_1}^{x_2} F\,dx$ on which a given constraint $I_2 = \int_{x_1}^{x_2} G\,dx = C$ (with C being a constant) is imposed is extremized by the function $y(x)$ *iff* the functional $I[y] = I_1 + \lambda I_2$ is extremized by the function $y(x)$. So, instead of extremizing I_1 in our search we extremize I, and hence the functional of Eq. 1 becomes:

$$\begin{aligned} I[y] &= \int_{x_1}^{x_2} F(x, y, y')\,dx + \lambda \int_{x_1}^{x_2} G(x, y, y')\,dx \\ &= \int_{x_1}^{x_2} \Big[F(x, y, y') + \lambda G(x, y, y')\Big]\,dx \end{aligned}$$

---

[29] In fact, "extremum" in the above statement represents the issue of interest in the calculus (or mathematics) of variations (otherwise "stationary" is more inclusive). Also, whether we should use *if* or *iff* in the above statement depends in our opinion on how we view the problem (i.e. whether it is a variational problem or a constrained variational problem).

$$= \int_{x_1}^{x_2} H\left(x, y, y'\right)\, dx \tag{21}$$

where $H \equiv F + \lambda G$. Accordingly, the Euler-Lagrange equation (see Eq. 2) becomes:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \tag{22}$$

Again, more than one constraint can be embedded using more than one Lagrange multiplier. Therefore, if we have n constraints then the variational formulation will take the form $F + \lambda_1 G_1 + \cdots + \lambda_n G_n$, and hence we use the integrand $H \equiv F + \lambda_1 G_1 + \cdots + \lambda_n G_n$ (instead of F) in our functional integral with the employment of Eq. 22 (instead of Eq. 2).

Regarding the solution of this type of problems, we note that the Euler-Lagrange equation is a second order differential equation and hence the solution y usually contains two constants of integration. Moreover, it contains n Lagrange multipliers λ's (where n is the number of the given constraints). Hence, the obtained solution usually contains $n + 2$ unknowns. Accordingly, to obtain a specific solution we should use the two boundary conditions at the end points plus the n constraining conditions $\int_{x_1}^{x_2} G_i\, dx = C_i$ $(i = 1, \cdots, n)$. This will be applied and clarified in some of the upcoming Problems.[30]

We should finally note that although the Lagrange multiplier(s) λ enter in the formulation of variational problems with constraints, in many cases the determination of λ is irrelevant to the required solution (even though the determination of λ may be needed provisionally in some of these cases).[31] Accordingly, λ (or λ's) is just a tool to obtain the solution and hence the reader should not worry about λ and its value or nature if the sought solution is obtained. Yes, there are some types of problems in which λ (or λ's) has certain mathematical or physical significance and hence the determination of λ may be desirable or even indispensable. These issues will be seen (and investigated in practical terms) in the upcoming Problems which are solved by this technique (or they are linked to this technique). We should also note that the Lagrange multipliers technique does not necessarily lead to extremization (since it is essentially a stationarization technique which is more general) although we were generally talking above about extremization (which is the main purpose of the variational techniques). So, further investigation to determine the nature of the obtained solution (i.e. minimum or maximum or inflection or saddle) may be required.

Problems

1. Justify the logic of the Lagrange multipliers technique using a simple argument.
   Answer: Starting from Eq. 21 we have:

   $$I = \int_{x_1}^{x_2} H\, dx = \int_{x_1}^{x_2} \left(F + \lambda G\right)\, dx = \int_{x_1}^{x_2} F\, dx + \lambda \int_{x_1}^{x_2} G\, dx = I_1 + \lambda I_2$$

   So, if I_1 is extremum (or stationary) and I_2 is constant (according to the constraint) then I should also be extremum (or stationary).
2. Outline the procedure of the Lagrange multipliers technique in solving constrained variational problems.
   Answer:[32] If $f(x)$ is a function to be optimized (or stationarized) subject to a constraint $g(x) = c$ (with c being a constant), then according to the procedure of the Lagrange multipliers technique we do the following:
   - We define a new function $h = f + \lambda g$ where λ is an extremizing (or stationarizing) parameter. The

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[30] As indicated in the phrasing of this paragraph, we are talking about a typical problem of this type; otherwise some problems may not strictly follow the above description and procedure.

[31] In fact, this is behind the "Lagrange *undetermined* multiplier" that is used to label λ in some texts.

[32] This answer is rather generic and lacks rigorous technicalities. The purpose of it is to give a general idea about the Lagrange multipliers technique rather than a rigorous treatment and formulation. However, the technique will be more clarified by the many upcoming constrained variational Problems in this book which are formulated and solved by this technique.

function h is commonly known as the Lagrangian while the parameter λ is commonly known as the Lagrange multiplier (or Lagrange undetermined multiplier).

- We optimize h by taking its derivative with respect to x and equating the derivative to zero. The solution of this equation will yield the optimal point(s).
- If we have more than one constraint (say n constraints $g_1 = c_1, \cdots, g_n = c_n$) then h is defined as $h = f + \lambda_1 g_1 + \cdots + \lambda_n g_n$ and the above procedure (of taking derivative and equating it to zero) is repeated.
- If f and g (or g's) are multi-variable functions[33] (say they are functions of $x_1, \cdots, x_m$) then we take the partial derivatives of h with respect to these variables and equate the derivatives to zero (i.e. $\frac{\partial h}{\partial x_1} = 0, \cdots, \frac{\partial h}{\partial x_m} = 0$). The solution of this set of simultaneous equations will yield the optimal point(s).
- In the case of optimizing a functional (rather than a function) which is usually dealt with by the techniques of the calculus of variations, the above procedure is amended to cope with this altered situation. So, in the calculus of variations (where the functional is an integral) the function H (which is the integrand of the functional integral where $H = F + \lambda G$ or $H = F + \lambda_1 G_1 + \cdots + \lambda_n G_n$) is used as an input to the Euler-Lagrange equation (see Eq. 22) whose solution will yield the optimal function(s). Although this procedure looks rather different from the above-described procedure of the Lagrange multipliers technique, it essentially rests on the same logic and rationale.

3. We have a 3 meter rope which we want to shape into a rectangle such that the enclosed area is optimal (i.e. maximum). Find the dimensions of the required rectangle.
Answer: This is a constrained variational problem where we want to optimize a function (which is the area σ of the rectangle) subject to the constraint that the perimeter p of the rectangle is equal to 3. Now, if the lengths of the two sides of the rectangle are x and y then $f(x, y) \equiv \sigma = xy$ and $g(x, y) \equiv p = 2(x + y) = 3$ and hence $h = xy + \lambda 2(x + y)$. Accordingly:

$$\begin{aligned} \frac{\partial h}{\partial x} &= y + \lambda 2 = 0 \\ \frac{\partial h}{\partial y} &= x + \lambda 2 = 0 \end{aligned}$$

On subtracting the second equation from the first we get $y - x = 0$ and hence $y = x$ which means that our rectangle is a square of side length $x = 3/4$.
Note: we may use a single variable approach for solving this Problem where $f(x) \equiv \sigma = x(1.5 - x) = 1.5x - x^2$ and $g(x) \equiv p = C = 3$ (where C is a constant) and hence $h = f + \lambda g = 1.5x - x^2 + \lambda C$. Accordingly:

$$\frac{dh}{dx} = 1.5 - 2x = 0$$

which leads to the same solution, i.e. $x = 3/4$. However, this in reality is a non-constrained approach, i.e. the constraint is actually embedded in the formulation of f rather than being imposed as an additional condition in the formulation.

4. What is the Lagrange multipliers formulation for a functional $\int F$ that to be optimized subject to n constraints $\int G_1, \cdots, \int G_n$.
Answer: The formulation is given by Eqs. 21 and 22 with H being given by:

$$H = F + \sum_{i=1}^{n} \lambda_i G_i$$

5. Find the Euler-Lagrange equations for the following constrained variational problems:
(**a**) $F(x, y, y') = xy'^2$ with $G = x^2 y^2$.
(**b**) $F(x, y, y') = yy'^3$ with $G = xy$.

---

[33] We note that the Lagrange multipliers technique is generally associated with multi-variable functions (rather than single-variable functions). In fact, even when f is originally a single-variable function the Lagrange multipliers technique usually leads to multi-variable formulation.

(**c**) $F(x, y, y') = y/y'$ with $G = ax^2 + b$ (a and b are constants).
(**d**) $F(t, x, \dot{x}) = x$ with $G = \alpha\dot{x}^2$ (α is a constant).
Answer:
(**a**) We use Eq. 22 with $H \equiv F + \lambda G = xy'^2 + \lambda x^2 y^2$ and hence the Euler-Lagrange equation is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[xy'^2 + \lambda x^2 y^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[xy'^2 + \lambda x^2 y^2\right]\right) &= 0 \\
2\lambda x^2 y - \frac{d}{dx}(2xy') &= 0 \\
2\lambda x^2 y - 2y' - 2xy'' &= 0 \\
xy'' + y' - \lambda x^2 y &= 0
\end{aligned}$$

(**b**) We use Eq. 22 with $H \equiv F + \lambda G = yy'^3 + \lambda xy$ and hence the Euler-Lagrange equation is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[yy'^3 + \lambda xy\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[yy'^3 + \lambda xy\right]\right) &= 0 \\
y'^3 + \lambda x - \frac{d}{dx}\left(3yy'^2\right) &= 0 \\
y'^3 + \lambda x - 3y'^3 - 6yy'y'' &= 0 \\
6yy'y'' + 2y'^3 - \lambda x &= 0
\end{aligned}$$

(**c**) We use Eq. 22 with $H \equiv F + \lambda G = \frac{y}{y'} + \lambda a x^2 + \lambda b$ and hence the Euler-Lagrange equation is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[\frac{y}{y'} + \lambda a x^2 + \lambda b\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[\frac{y}{y'} + \lambda a x^2 + \lambda b\right]\right) &= 0 \\
\frac{1}{y'} - \frac{d}{dx}\left(-\frac{y}{y'^2}\right) &= 0 \\
\frac{1}{y'} + \frac{y'}{y'^2} - \frac{2yy''}{y'^3} &= 0 \\
\frac{2}{y'} - \frac{2yy''}{y'^3} &= 0 \\
\frac{yy''}{y'^3} - \frac{1}{y'} &= 0
\end{aligned}$$

(**d**) We use Eq. 22 with $H \equiv F + \lambda G = x + \lambda\alpha\dot{x}^2$ and hence the Euler-Lagrange equation (noting that x, y, y' in Eq. 22 correspond to $t, x, \dot{x}$ here) is:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[x + \lambda\alpha\dot{x}^2\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left[x + \lambda\alpha\dot{x}^2\right]\right) &= 0 \\
1 - \frac{d}{dt}(2\lambda\alpha\dot{x}) &= 0 \\
1 - 2\lambda\alpha\ddot{x} &= 0 \\
\ddot{x} - \frac{1}{2\lambda\alpha} &= 0
\end{aligned}$$

6. Find and solve the Euler-Lagrange equations for the following constrained variational problems:
(**a**) $F(x, y, y') = y$ and $G = \sqrt{1 + y'^2}$.
(**b**) $F(x, y, y') = y'^{1/2}$ and $G = x^2$.
Answer:
(**a**) We use Eq. 22 with $H \equiv F + \lambda G = y + \lambda\sqrt{1 + y'^2}$, that is:

$$\frac{\partial}{\partial y}\left[y + \lambda\sqrt{1 + y'^2}\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y + \lambda\sqrt{1 + y'^2}\right]\right) = 0$$

$$\begin{aligned} 1-\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) &= 0 \\ \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) &= 1 \end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned} \frac{\lambda y'}{\sqrt{1+y'^2}} &= x+C \\ \lambda^2 y'^2 &= (x+C)^2\left(1+y'^2\right) \\ y'^2 &= \frac{(x+C)^2}{\lambda^2-(x+C)^2} \\ y' &= \pm\frac{(x+C)}{\sqrt{\lambda^2-(x+C)^2}} \\ y &= \mp\sqrt{\lambda^2-(x+C)^2}+D \\ (x+C)^2+(y-D)^2 &= \lambda^2 \end{aligned}$$

(**b**) $H \equiv F+\lambda G = y'^{1/2}+\lambda x^2$ which is independent of y and hence we can use Eq. 4 (with H replacing F), that is:

$$\begin{aligned} \frac{\partial}{\partial y'}\left(y'^{1/2}+\lambda x^2\right) &= C \\ \frac{1}{2y'^{1/2}} &= C \end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned} y' &= \frac{1}{4C^2} \\ y &= \frac{x}{4C^2}+D \end{aligned}$$

7. Find and solve the Euler-Lagrange equations for the following constrained variational problems (subject to the given boundary conditions and constraints):
(**a**) $F(x,y,y') = y'^2 - y^2$ and $G = y$ with $y(x=0)=0$, $y(x=\pi)=1$ and $\int_0^\pi G\,dx = 1$.
(**b**) $F(x,y,y') = y'^2$ and $G = -y^2$ with $y(x=0)=0$ and $y'(x=1)=0$.
Answer:
(**a**) We use Eq. 22 with $H \equiv F+\lambda G = y'^2 - y^2 + \lambda y$, that is:

$$\begin{aligned} \frac{\partial}{\partial y}\left[y'^2-y^2+\lambda y\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2-y^2+\lambda y\right]\right) &= 0 \\ -2y+\lambda-\frac{d}{dx}\left(2y'\right) &= 0 \\ -2y+\lambda-2y'' &= 0 \\ y''+y-\frac{\lambda}{2} &= 0 \end{aligned}$$

So, the solution is $y = C\cos x + D\sin x + \frac{\lambda}{2}$ which can be checked by substitution, that is:

$$\left[C\cos x+D\sin x+\frac{\lambda}{2}\right]''+\left[C\cos x+D\sin x+\frac{\lambda}{2}\right]-\frac{\lambda}{2} \overset{?}{=} 0$$

$$\begin{aligned}
\big[-C\sin x + D\cos x + 0\big]' + \left[C\cos x + D\sin x + \frac{\lambda}{2}\right] - \frac{\lambda}{2} &\overset{?}{=} 0\\
\big[-C\cos x - D\sin x\big] + \left[C\cos x + D\sin x + \frac{\lambda}{2}\right] - \frac{\lambda}{2} &\overset{?}{=} 0\\
0 &= 0
\end{aligned}$$

Now, from the condition $y\,(x=0)=0$ we get $C+\frac{\lambda}{2}=0$ and hence $C=-\frac{\lambda}{2}$ while from the condition $y(x=\pi)=1$ we get $-\frac{\lambda}{2}(-1)+0+\frac{\lambda}{2}=1$ and hence $\lambda=1$. Thus, the solution becomes $y=-\frac{1}{2}\cos x+D\sin x+\frac{1}{2}$. Also, from the constraint we get:

$$\begin{aligned}
\int_0^\pi \left(-\frac{1}{2}\cos x + D\sin x + \frac{1}{2}\right) dx &= 1\\
\int_0^\pi \left(-\cos x + 2D\sin x + 1\right) dx &= 2\\
\left[-\sin x - 2D\cos x + x\right]_0^\pi &= 2\\
\left[-0-2D\,(-1)+\pi\right] - \left[-0-2D+0\right] &= 2\\
2D+\pi+2D &= 2\\
4D+\pi &= 2\\
D &= \frac{2-\pi}{4}
\end{aligned}$$

Therefore, the solution is:

$$y = -\frac{1}{2}\cos x + \left(\frac{2-\pi}{4}\right)\sin x + \frac{1}{2}$$

(**b**) We use Eq. 22 with $H \equiv F + \lambda G = y'^2 - \lambda y^2$, that is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y'^2-\lambda y^2\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2-\lambda y^2\right]\right) &= 0\\
-2\lambda y - \frac{d}{dx}\left(2y'\right) &= 0\\
y''+\lambda y &= 0
\end{aligned}$$

Now:
If $\lambda<0$ then $y=a\cosh\left(\sqrt{|\lambda|}x\right)+b\sinh\left(\sqrt{|\lambda|}x\right)$ $\big[$with a,b being constants$\big]$ and from the first boundary condition we have $0=a\cosh(0)+b\sinh(0)$ $\big[$and hence $a=0\big]$ while from the second boundary condition we have $0=b\sqrt{|\lambda|}\cosh\left(\sqrt{|\lambda|}\right)$ $\big[$and hence $b=0$ because $\cosh\left(\sqrt{|\lambda|}\right)=0$ has no real solution$\big]$. So, the solution is $y=0$.
If $\lambda=0$ then $y=ax+b$ (with a,b being constants) and from the first boundary condition we have $0=a0+b$ (and hence $b=0$) while from the second boundary condition we have $0=a$. So, the solution is $y=0$.[34]
If $\lambda>0$ then $y=a\cos\left(\sqrt{\lambda}x\right)+b\sin\left(\sqrt{\lambda}x\right)$ $\big[$with a,b being constants$\big]$ and from the first boundary condition we have $0=a\cos(0)+b\sin(0)$ $\big[$and hence $a=0\big]$ while from the second boundary condition we have $0=b\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right)$ $\big[$and hence if we assume non-trivial solution, i.e. $b\neq0$, then $\sqrt{\lambda}=\pi/2\big]$. So, the solution is $y=b\sin\left(\frac{\pi x}{2}\right)$.[35]

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[34] We consider the case $\lambda=0$ for the sake of comprehensiveness (considering more general situations); otherwise the multiplier λ is not supposed to be zero.

[35] In fact, this is the principal solution; otherwise $\sqrt{\lambda}$ can be an odd multiple of $\pi/2$.

8. Obtain a specific solution for part (b) of Problem 7 assuming $y > 0$ plus the following constraint condition: $\int_0^1 G\,dx = -\pi$.
Answer: From the given constraint we get:

$$\begin{aligned}
\int_0^1 -b^2 \sin^2\left(\frac{\pi x}{2}\right) dx &= -\pi \\
-\frac{b^2}{2}\left[x - \frac{1}{\pi}\sin(\pi x)\right]_0^1 &= -\pi \\
-\frac{b^2}{2}(1 - 0 - 0 + 0) &= -\pi \\
b &= \pm\sqrt{2\pi}
\end{aligned}$$

Therefore, the specific solution (taking the positive root only since $y > 0$) is:

$$y = \sqrt{2\pi}\sin\left(\frac{\pi x}{2}\right) \qquad (0 \le x \le 1)$$

9. Find the Euler-Lagrange equations for the following constrained variational problems:
(**a**) $F(x, y, y') = x^2 y'^2$ with $G_1 = xy$ and $G_2 = y\sqrt{y'}$.
(**b**) $F(x, y, y') = y'^{1/2}$ with $G_1 = y^2$ and $G_2 = ax^3$ (a is constant).
Answer:
(**a**) We use Eq. 22 with $H \equiv F + \lambda_1 G_1 + \lambda_2 G_2 = x^2 y'^2 + \lambda_1 xy + \lambda_2 y\sqrt{y'}$ and hence the Euler-Lagrange equation is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[x^2 y'^2 + \lambda_1 xy + \lambda_2 y\sqrt{y'}\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[x^2 y'^2 + \lambda_1 xy + \lambda_2 y\sqrt{y'}\right]\right) &= 0 \\
\lambda_1 x + \lambda_2\sqrt{y'} - \frac{d}{dx}\left(2x^2 y' + \lambda_2 \frac{y}{2\sqrt{y'}}\right) &= 0 \\
\lambda_1 x + \lambda_2\sqrt{y'} - 4xy' - 2x^2 y'' - \lambda_2 \frac{y'}{2\sqrt{y'}} + \lambda_2 \frac{yy''}{4y'^{3/2}} &= 0 \\
\lambda_1 x + \lambda_2\sqrt{y'} - 4xy' - 2x^2 y'' - \frac{\lambda_2}{2}\sqrt{y'} + \frac{\lambda_2}{4}\frac{yy''}{y'^{3/2}} &= 0 \\
\lambda_1 x + \frac{\lambda_2}{2}\sqrt{y'} - 4xy' - 2x^2 y'' + \frac{\lambda_2}{4}\frac{yy''}{y'^{3/2}} &= 0
\end{aligned}$$

(**b**) We use Eq. 22 with $H \equiv F + \lambda_1 G_1 + \lambda_2 G_2 = y'^{1/2} + \lambda_1 y^2 + \lambda_2 ax^3$ and hence the Euler-Lagrange equation is:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[y'^{1/2} + \lambda_1 y^2 + \lambda_2 ax^3\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^{1/2} + \lambda_1 y^2 + \lambda_2 ax^3\right]\right) &= 0 \\
2\lambda_1 y - \frac{d}{dx}\left(\frac{1}{2y'^{1/2}}\right) &= 0 \\
2\lambda_1 y - \left(-\frac{y''}{4y'^{3/2}}\right) &= 0 \\
2\lambda_1 y + \frac{y''}{4y'^{3/2}} &= 0 \\
y'' + 8\lambda_1 y y'^{3/2} &= 0
\end{aligned}$$

1.9 Variational Problems with Variable Boundaries

So far, the variational problems that we dealt with have fixed boundaries. In some variational problems, the boundary is partially or totally variable. In fact, there are many cases and details in this regard.

So, in this section we briefly investigate only some simple cases of variational problems with variable boundaries.[36]

The simplest of these cases is when the variational problems (of single variable) have only one fixed end point and hence the other end point is variable. In this case the Euler-Lagrange equation (i.e. Eq. 2) will be used as before but with a modification to the boundary conditions where a transversality condition will be imposed on the variable end point (while the fixed end point keeps its fixed boundary condition as before). To be more specific, let have a curve Γ represented by a function $y_\Gamma = y(x)$ and it connects a fixed point A to a curve Γ_1 where the end point B of Γ is restricted to move freely on Γ_1 (as depicted in Figure 4). In this case, the transversality condition states: if the value of the functional $I\left[y_\Gamma\right] = \int_{x_1}^{x_2} F\left(x, y, y'\right) dx$ is optimal (with respect to neighboring curves connecting point A to curve Γ_1) then at point B the direction $dX : dY$ of Γ_1 and the element of Γ should satisfy the relation:

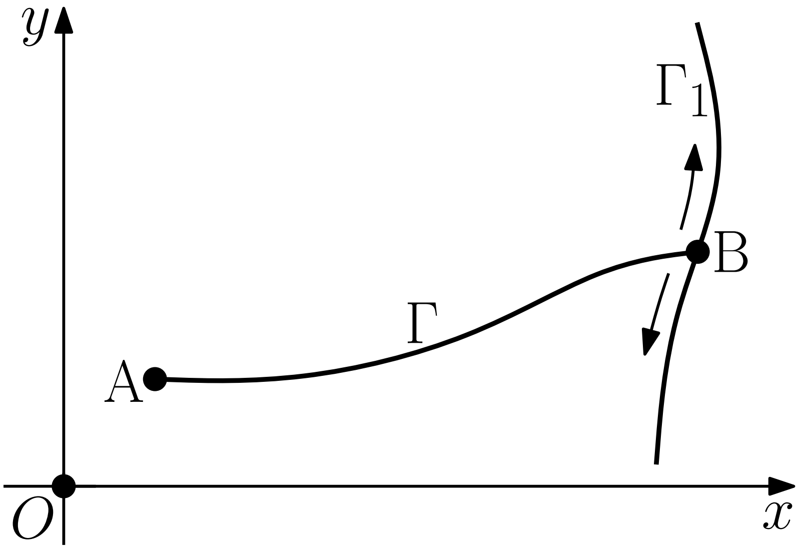

Figure 4: A schematic illustration of the setting of variational problems with variable boundaries in the simplest case when a curve Γ (that is to be optimized) connects a fixed point A to a given boundary curve Γ_1 where the end point B of Γ is restricted to move freely on Γ_1. See § 1.9.

$$F\,dX + \left(dY - y'dX\right) F_{y'} = 0 \qquad \text{(at point B)} \qquad (23)$$

where y' belongs to Γ at B while dX and dY belong to Γ_1 at B, and $F_{y'}$ is the partial derivative of F with respect to y'.

Another case is when a curve Γ represented by a function $y_\Gamma = y(x)$ is connecting two other curves (Γ_1 and Γ_2) where the end points (B_1 and B_2) of Γ are restricted to move freely on Γ_1 and Γ_2 (as depicted in Figure 5). In this case, the transversality condition states: if the value of the functional $I\left[y_\Gamma\right]$ is optimal (with respect to neighboring curves connecting Γ_1 and Γ_2) then at the points B_1 and B_2 the directions $dX_1 : dY_1$ of Γ_1 and $dX_2 : dY_2$ of Γ_2 and the corresponding elements of Γ should satisfy the following two relations:

$$F\,dX_1 + \left(dY_1 - y'dX_1\right) F_{y'} = 0 \qquad \text{(at point B}_1\text{)} \qquad (24)$$

$$F\,dX_2 + \left(dY_2 - y'dX_2\right) F_{y'} = 0 \qquad \text{(at point B}_2\text{)} \qquad (25)$$

where again y' belongs to Γ at B_1 and B_2 while dX_1 and dY_1 belong to Γ_1 at B_1 and dX_2 and dY_2 belong to Γ_2 at B_2.

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[36] Variable boundaries may also be called free boundaries because free movement within certain restrictions is allowed. However, in our view "variable" is more appropriate than "free" because it is not really free.

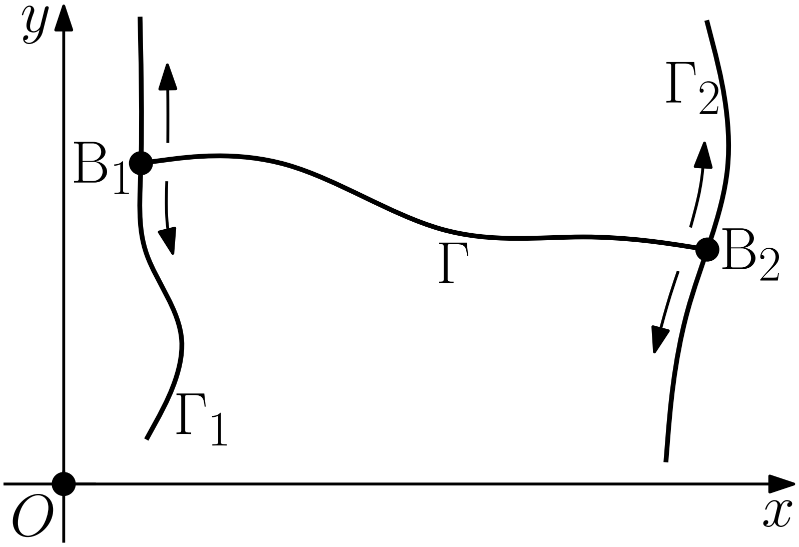

Figure 5: A schematic illustration of the setting of variational problems with variable boundaries in the case when a curve Γ (that is to be optimized) connects two given boundary curves (Γ_1 and Γ_2) where the end points (B_1 and B_2) of Γ are restricted to move freely on Γ_1 and Γ_2. See § 1.9.

Finally, we should draw the attention to the following important remarks:

- Although in many cases the transversality conditions lead to the perpendicularity of the extremal curve and the boundary curve (i.e. $\Gamma \perp \Gamma_1$ or $\Gamma \perp \Gamma_2$), this is not necessarily the case and hence in the variational problems with variable boundaries the transversality conditions may not lead to orthogonality.
- To avoid potential confusion, the reader should note that the transversality condition may be given in some texts by other forms like the following:

$$(F - y'F_{y'})\, dX + F_{y'} dY = 0 \tag{26}$$

which is identical to Eq. 23 but with different ordering and grouping of terms. The transversality equation may also be "divided" by dX and hence Eq. 23 (as well as Eqs. 24 and 25) becomes $F + (Y' - y')\, F_{y'} = 0$ where $Y' = dY/dX$ (and similarly Eq. 26 becomes $F - y'F_{y'} + F_{y'}Y' = 0$).

Problems

1. Solve the following variational problems (in which we have one fixed boundary and one variable boundary):
   (**a**) $F(x, y, y') = y'^2 + 2yy' - y^2$ with fixed boundary $y(0) = 1$ and variable boundary $x = \pi/4$.
   (**b**) $F(x, y, y') = y'^2 - xy'$ with fixed boundary $y(0) = 5$ and variable boundary $x = 3$.
   (**c**) $F(x, y, y') = \sqrt{1 + y'^2}/y$ with fixed boundary $y(0) = 0$ and variable boundary $y = x + 2$.
   Answer:
   (**a**) In this case we have one point of the curve fixed by the condition $y(0) = 1$ while the other point of the curve is free to move on the vertical line $x = \pi/4$. In part (j) of Problem 11 of § 1.4 we solved this Problem (without boundary conditions) and obtained the solution $y = a \sin x + b \cos x$. Now, from the fixed boundary we get $b = 1$ and hence $y = a \sin x + \cos x$. Regarding the variable boundary, we impose the transversality condition:

$$F\, dX + (dY - y'dX)\, F_{y'} = 0$$

   Now, since the boundary curve $x = \pi/4$ is a vertical line then $dX = 0$ (noting that in the transversality condition y' belongs to the extremal curve while dX and dY belong to the boundary curve) and hence the transversality condition becomes $dYF_{y'} = 0$. If we now note that on a vertical line $dY \neq 0$ then the transversality condition will be reduced to:

$$\begin{aligned}
F_{y'} &= 0 \\
2y' + 2y &= 0 \\
y' + y &= 0 \\
(a\cos x - \sin x) + (a\sin x + \cos x) &= 0 \\
(a+1)\cos x + (a-1)\sin x &= 0 \\
\frac{a+1}{1-a} &= \tan x \\
\frac{a+1}{1-a} &= 1 \qquad\qquad (x = \pi/4) \\
a &= 0
\end{aligned}$$

So, the solution is $y = \cos x$ (which is plotted in Figure 6). The two curves meet at point $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$.

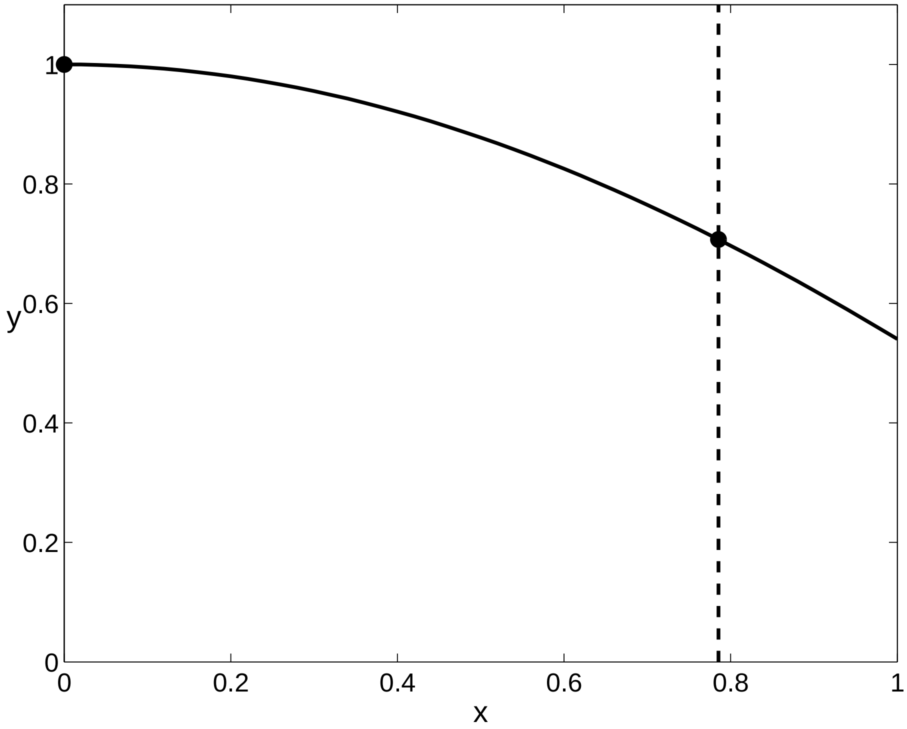

Figure 6: Plot of the extremal curve $y = \cos x$ (solid) and the boundary line $x = \pi/4$ (dashed). See part (a) of Problem 1 of § 1.9.

(**b**) We have $F = y'^2 - xy'$ and hence the Euler-Lagrange equation (noting that y is missing in F and hence we use Eq. 4) is:

$$\begin{aligned}
\frac{\partial}{\partial y'}\left(y'^2 - xy'\right) &= C \\
2y' - x &= C \\
y' &= \frac{x}{2} + \frac{C}{2}
\end{aligned}$$

On integrating this equation once we get:

$$y = \frac{x^2}{4} + \frac{C}{2}x + D$$

Now, from the fixed boundary $y(0) = 5$ we get $D = 5$ and hence $y = \frac{x^2}{4} + \frac{C}{2}x + 5$. Regarding the variable boundary, we follow similar procedures and arguments to those of part (a) of this Problem and hence we get the following transversality condition:

$$\begin{aligned} F_{y'} &= 0 \\ 2y' - x &= 0 \\ y' &= \frac{x}{2} \\ \frac{x}{2} + \frac{C}{2} &= \frac{x}{2} \\ C &= 0 \end{aligned}$$

Accordingly, the solution is $y = \frac{x^2}{4} + 5$ (which is plotted in Figure 7). The two curves meet at point $(3, 7.25)$.

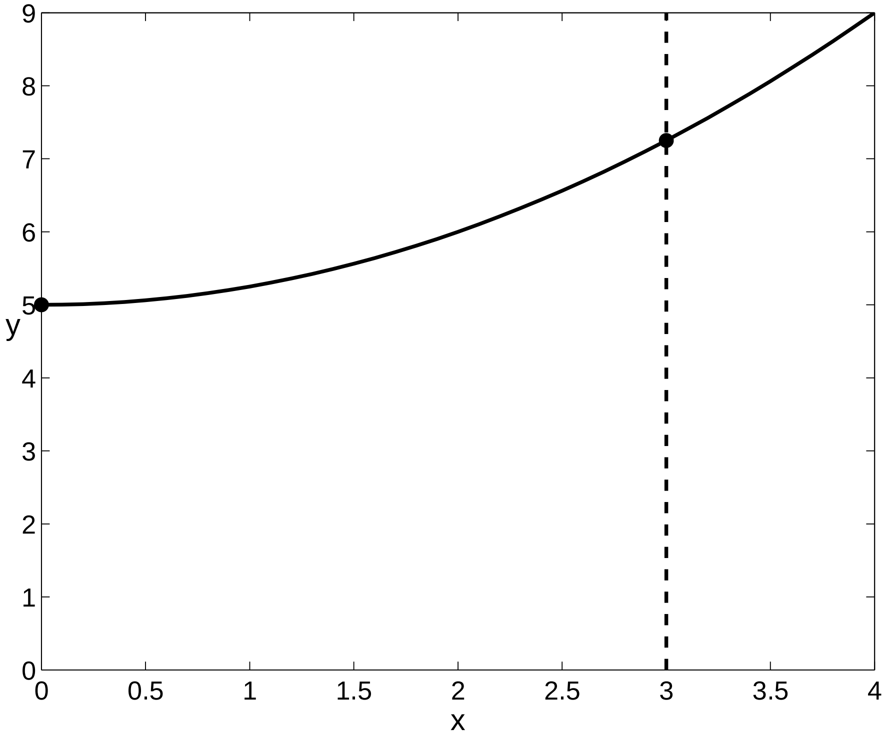

Figure 7: Plot of the extremal curve $y = \frac{x^2}{4} + 5$ (solid) and the boundary line $x = 3$ (dashed). See part (b) of Problem 1 of § 1.9.

(**c**) We have $F = \sqrt{1 + y'^2}/y$ and hence the Euler-Lagrange equation (noting that x is missing in F and hence we use Eq. 3) is:

$$\left[\frac{\sqrt{1+y'^2}}{y}\right] - y'\frac{\partial}{\partial y'}\left[\frac{\sqrt{1+y'^2}}{y}\right] = C$$

$$\begin{aligned}
\frac{\sqrt{1+y'^2}}{y} - y'\left[\frac{y'}{y\sqrt{1+y'^2}}\right] &= C \\
\frac{\sqrt{1+y'^2}}{y} - \frac{y'^2}{y\sqrt{1+y'^2}} &= C \\
\frac{1+y'^2}{y\sqrt{1+y'^2}} - \frac{y'^2}{y\sqrt{1+y'^2}} &= C \\
\frac{1}{y\sqrt{1+y'^2}} &= C \\
y^2\left(1+y'^2\right) &= \frac{1}{C^2}
\end{aligned}$$

This is the Euler-Lagrange equation which we solve as follows:

$$\begin{aligned}
y'^2 &= \frac{1-C^2y^2}{C^2y^2} \\
y' &= \pm\frac{\sqrt{1-C^2y^2}}{Cy} \\
\frac{Cy}{\sqrt{1-C^2y^2}}\,dy &= \pm dx \\
-\frac{\sqrt{1-C^2y^2}}{C} &= \pm x + C_1
\end{aligned}$$

Now, from the fixed condition $y(0) = 0$ we get $C_1 = -1/C$ and hence:

$$\begin{aligned}
-\frac{\sqrt{1-C^2y^2}}{C} &= \pm x - \frac{1}{C} \\
\sqrt{1-C^2y^2} &= \mp Cx + 1 \\
1 - C^2y^2 &= C^2x^2 \mp 2Cx + 1 \\
-C^2y^2 &= C^2x^2 \mp 2Cx \\
y^2 &= -x^2 \pm \frac{2}{C}x \\
y^2 &= -x^2 + bx \qquad (b = \pm 2/C) \qquad (27)
\end{aligned}$$

Regarding the variable boundary, we impose the transversality condition:

$$\begin{aligned}
F\,dX + \left(dY - y'dX\right)F_{y'} &= 0 \\
F + \left(\frac{dY}{dX} - y'\right)F_{y'} &= 0 \\
F + \left(1 - y'\right)F_{y'} &= 0 \qquad \text{(the boundary curve is } Y = X + 2\text{)} \\
\frac{\sqrt{1+y'^2}}{y} + \left(1-y'\right)\left(\frac{y'}{y\sqrt{1+y'^2}}\right) &= 0 \\
\frac{1+y'^2}{y\sqrt{1+y'^2}} + \frac{y'}{y\sqrt{1+y'^2}} - \frac{y'^2}{y\sqrt{1+y'^2}} &= 0 \\
\frac{1+y'}{y\sqrt{1+y'^2}} &= 0 \\
1 + y' &= 0 \\
y' &= -1
\end{aligned}$$

Referring to Eq. 27, the extremal curve is given by $y^2 = -x^2 + bx$. On differentiating this implicitly we get $2yy' = -2x + b$. Now, on the point of intersection (where the extremal curve and the boundary curve meet) we should also have $y = x + 2$ (since this point is on the boundary curve) as well as $y' = -1$ and hence on substituting from $y = x + 2$ and $y' = -1$ into $2yy' = -2x + b$ we get:

$$\begin{aligned} 2(x+2)(-1) &= -2x + b \\ -2x - 4 &= -2x + b \\ b &= -4 \end{aligned}$$

Accordingly, the solution (taking the positive root) is $y = \sqrt{-x^2 - 4x}$ (which is plotted in Figure 8). The two curves meet at point $(\sqrt{2} - 2, \sqrt{2})$.

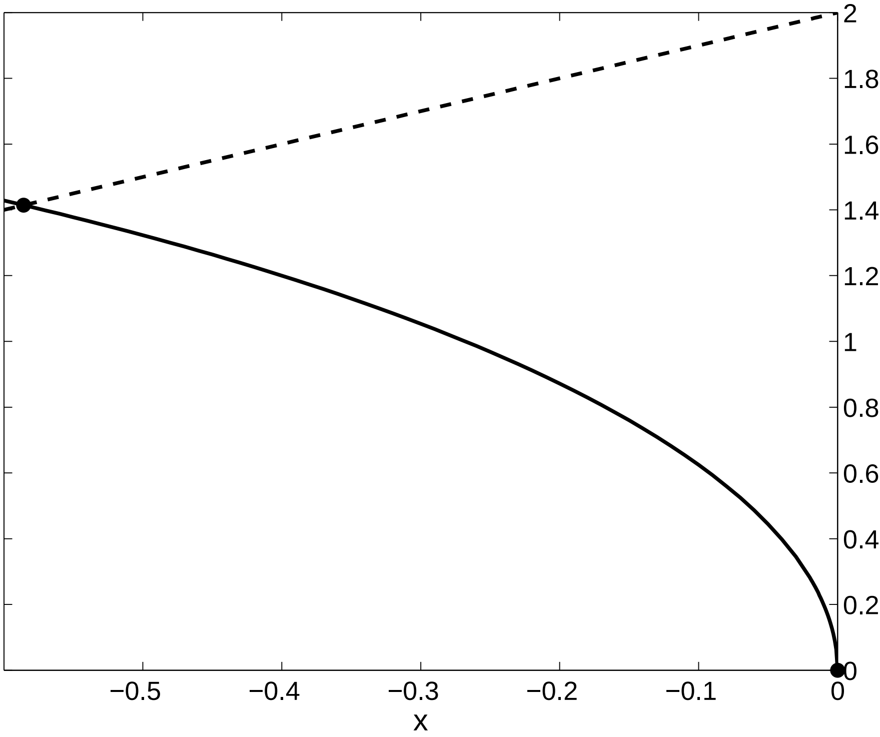

Figure 8: Plot of the extremal curve $y = \sqrt{-x^2 - 4x}$ (solid) and the boundary line $y = x + 2$ (dashed). See part (c) of Problem 1 of § 1.9.

2. Solve the following variational problems (in which we have two variable boundaries):
   (**a**) $F(x, y, y') = y'^2 + y' + yy' + 2y$ with $x = 0$ and $x = 2$.
   (**b**) $F(x, y, y') = \sqrt{1 + y'^2}$ with $y = x^4$ and $y = x - 1$.
   Answer:
   (**a**) In this case the left point of the extremal curve is free to move on the vertical line $x = 0$ while the right point of the extremal curve is free to move on the vertical line $x = 2$. The Euler-Lagrange equation is:

$$\begin{aligned} \frac{\partial}{\partial y}\left[y'^2 + y' + yy' + 2y\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y'^2 + y' + yy' + 2y\right]\right) &= 0 \\ y' + 2 - \frac{d}{dx}(2y' + 1 + y) &= 0 \\ y' + 2 - 2y'' - y' &= 0 \end{aligned}$$

$$y'' - 1 = 0$$

So, the solution is $y = \frac{1}{2}x^2 + Cx + D$ (with C and D being constants). This can be easily verified by substitution into the last equation.
Following similar procedures and arguments to those of part (a) of Problem 1 we get the following transversality conditions:

$$F_{y'} = 0 \quad \text{at } x = 0 \qquad \text{and} \qquad F_{y'} = 0 \quad \text{at } x = 2$$

Accordingly, at $x = 0$ we have:

$$\begin{aligned}
\frac{\partial}{\partial y'}\left(y'^2 + y' + yy' + 2y\right) &= 0 \\
2y' + 1 + y &= 0 \\
2\left(x + C + 0\right) + 1 + \left(\frac{1}{2}x^2 + Cx + D\right) &= 0 \\
2\left(0 + C + 0\right) + 1 + \left(0 + 0 + D\right) &= 0 \qquad (x = 0) \\
2C + D + 1 &= 0
\end{aligned}$$

Similarly, at $x = 2$ we have:

$$\begin{aligned}
2\left(2 + C + 0\right) + 1 + \left(2 + 2C + D\right) &= 0 \qquad (x = 2) \\
4C + D + 7 &= 0
\end{aligned}$$

Hence, $C = -3$ and $D = 5$. So, the solution is $y = \frac{1}{2}x^2 - 3x + 5$ (which is plotted in Figure 9). The extremal curve meets with the boundary line $x = 0$ at point $(0, 5)$ and with the boundary line $x = 2$ at point $(2, 1)$.
(**b**) In this case one boundary point of the extremal curve is free to move on the curve $y = x^4$ while the other boundary point of the extremal curve is free to move on the line $y = x - 1$. Referring to part (i) of Problem 9 of § 1.4 (also see footnote [12]), the Euler-Lagrange equation for this Problem is $y' = a$ and hence $y = ax + b$ (with a and b being constants). Following similar procedures and arguments to those of the previous Problems we get the following transversality conditions:

$$\begin{aligned}
F + \left(\frac{dY}{dX} - y'\right) F_{y'} &= 0 \qquad \text{(for the boundary } Y = x^4\text{)} \\
\sqrt{1 + y'^2} + \left(4x^3 - y'\right) \frac{y'}{\sqrt{1 + y'^2}} &= 0 \\
1 + y'^2 + \left(4x^3 - y'\right) y' &= 0 \\
1 + 4x^3 y' &= 0 \\
1 + 4x^3 a &= 0 \qquad (28)
\end{aligned}$$

AND

$$\begin{aligned}
F + \left(\frac{dY}{dX} - y'\right) F_{y'} &= 0 \qquad \text{(for the boundary } Y = x - 1\text{)} \\
\sqrt{1 + y'^2} + \left(1 - y'\right) \frac{y'}{\sqrt{1 + y'^2}} &= 0 \\
1 + y'^2 + \left(1 - y'\right) y' &= 0 \\
1 + y' &= 0 \\
1 + a &= 0 \\
a &= -1 \qquad (29)
\end{aligned}$$

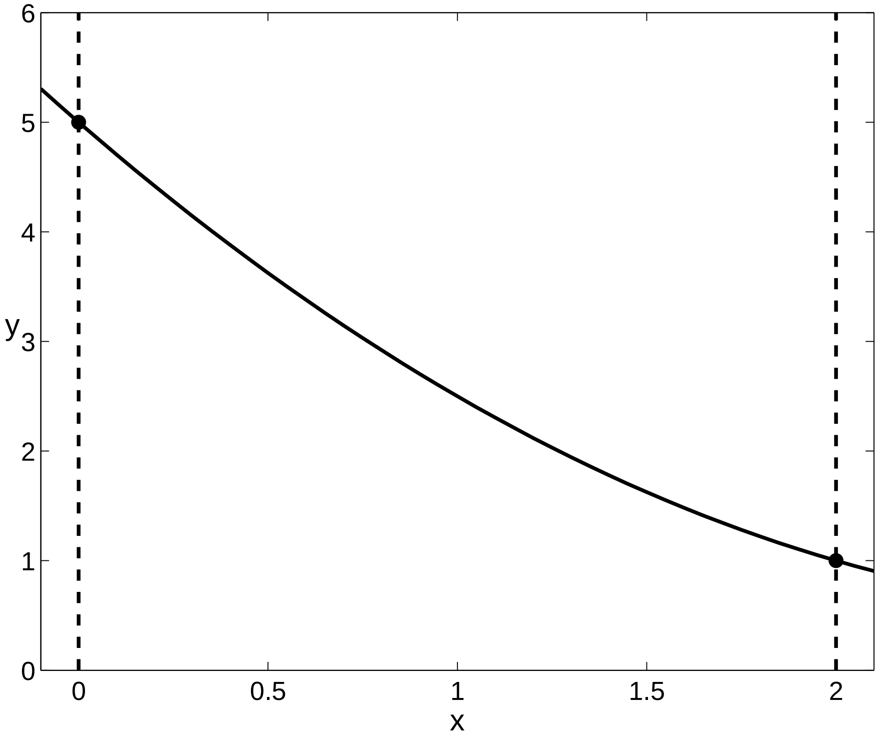

Figure 9: Plot of the extremal curve $y = \frac{1}{2}x^2 - 3x + 5$ (solid) and the boundary lines $x = 0$ and $x = 2$ (dashed). See part (a) of Problem 2 of § 1.9.

Therefore, the equation of the extremal curve becomes $y = -x + b$. So, what is left is to find b. On substituting from Eq. 29 into Eq. 28 we get $1 - 4x^3 = 0$ and hence $x = 4^{-1/3}$. Now, the point with coordinate $x = 4^{-1/3}$ is where the extremal curve $y = -x + b$ and the boundary curve $y = x^4$ meet and hence we should have:

$$\begin{aligned} -x + b &= x^4 \\ -4^{-1/3} + b &= 4^{-4/3} \\ b &= 4^{-4/3} + 4^{-1/3} \\ b &\simeq 0.787451 \end{aligned}$$

Accordingly, the solution is $y = -x + 0.787451$ (which is plotted in Figure 10). The extremal curve meets with the boundary curve $y = x^4$ at point $(4^{-1/3}, 4^{-4/3})$ and with the boundary line $y = x - 1$ at point $(0.893725, -0.106275)$.

3. Find the solution of the variational problem $I[y] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$ when we have:
(**a**) Fixed boundary $y(0) = 0$ and variable boundary $x = 1$.
(**b**) Fixed boundary $y(1) = 1$ and variable boundary $x = 0$.
(**c**) Variable boundaries $x = 0$ and $x = 1$.
Answer: Referring to part (i) of Problem 9 of § 1.4 (also see footnote [12]), the Euler-Lagrange equation for this Problem is $y' = a$ and hence $y = ax + b$ (with a and b being constants). Following similar procedures and arguments to those of the previous Problems we get the following transversality condition:

$$F_{y'} = \frac{y'}{\sqrt{1 + y'^2}} = 0 \qquad \text{and hence} \qquad y' = 0$$

So, the solution in all three cases (i.e. a, b and c) is $y = b$ (i.e. a horizontal line). Accordingly, all we

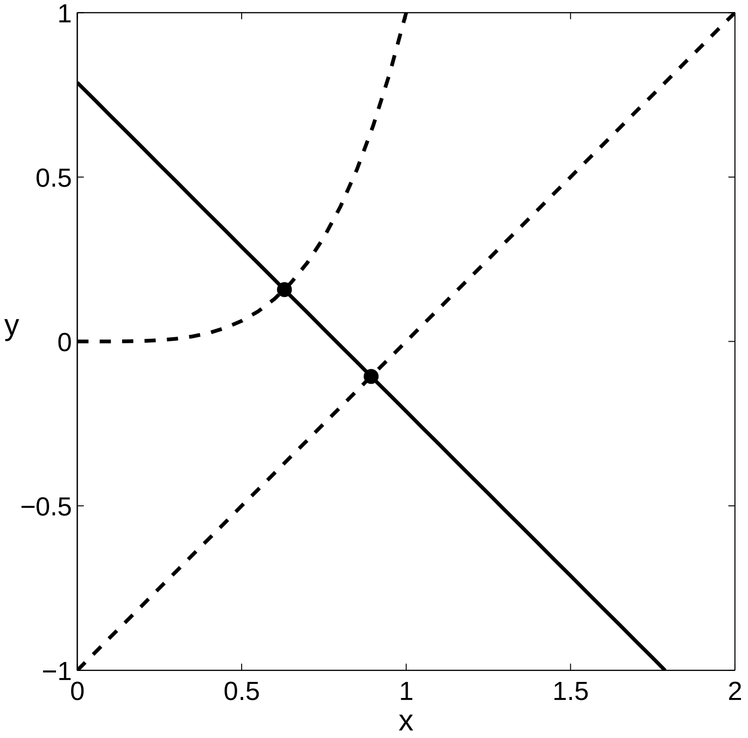

Figure 10: Plot of the extremal curve $y = -x + 0.787451$ (solid) and the boundary curves $y = x^4$ and $y = x - 1$ (dashed). See part (b) of Problem 2 of § 1.9.

need is to determine the value of b for each one of these cases, that is:
(**a**) From the fixed boundary condition $y(0) = 0$ we get $b = 0$ and hence the solution is $y = 0$.
(**b**) From the fixed boundary condition $y(1) = 1$ we get $b = 1$ and hence the solution is $y = 1$.
(**c**) There is no restriction on b (apart from being a constant) and hence the solution is $y = b$ (i.e. any horizontal line can be a solution).

4. Use the transversality condition in the following variational problems with variable boundaries to determine y' at the boundaries:[37]
(**a**) $F = y'^2 + y + C$ with boundary curves $x = 1$ and $x = 2$.
(**b**) $F = y'^2 + xy'$ with boundary curves $x = \alpha$ and $x = \beta$ (where α and β are constants).
(**c**) $F = y'^2 - 2yy'$ with boundary curves $x = a$ and $x = b$ (where a and b are constants).
Answer: The boundary curves in all these cases are vertical lines. So, referring to the previous Problems (see for instance part a of Problem 1) the transversality condition in all these cases should be given by $F_{y'} = 0$. Accordingly:
(**a**) In this case $F_{y'} = 2y' = 0$ and hence we should have $y' = 0$ at both boundaries, i.e. the slope of the extremal curve should vanish at the boundaries $x = 1$ and $x = 2$. So, the extremal curve at the boundary lines satisfies the following conditions: $y'(1) = y'(2) = 0$.
(**b**) In this case $F_{y'} = 2y' + x = 0$ and hence at the boundaries we should have $y' = -x/2$. So, the extremal curve at the boundary lines satisfies the following conditions: $y'(\alpha) = -\alpha/2$ and $y'(\beta) = -\beta/2$.
(**c**) In this case $F_{y'} = 2y' - 2y = 0$ and hence at the boundaries we should have $y' = y$. So, the extremal curve at the boundary lines satisfies the following conditions: $y'(a) = y(a)$ and $y'(b) = y(b)$.

---

[37] This sort of problems is usually studied within the context of investigating natural boundary conditions in the calculus of variations. However, because we have no appetite for going through these details we posed this Problem in this rather primitive way.

1.10 Variational Problems of Mixed Nature

In some variational problems we have more than one of the complications investigated in the previous sections (i.e. § 1.5 to § 1.9). For instance, we may have more than one independent variable and more than one dependent variable or we may have more than one dependent variable with some constraints. In such cases the aforementioned treatments apply simultaneously. For example, if the variational problem has m dependent variables $y_1, \cdots, y_m$ and n constraints $\int G_1, \cdots, \int G_n$ then the variational formulation is:[38]

$$I[y_1, \cdots, y_m] = \int_{x_1}^{x_2} H(x, y_1, \cdots, y_m, y'_1, \cdots, y'_m)\, dx$$

$$\frac{\partial H}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'_i}\right) = 0 \qquad (i = 1, \cdots, m)$$

where H is given by:

$$H = F + \sum_{i=1}^{n} \lambda_i G_i$$

Problems

1. State the variational formulation for a variational problem with two independent variables (α and β) and two dependent variables (y and z). Generalize this formulation to variational problems with m independent variables and n dependent variables.
Answer: Combining the formulations of § 1.6 and § 1.7, the functional integral is:

$$I[y, z] = \iint_\Omega F(\alpha, \beta, y, z, y_\alpha, z_\alpha, y_\beta, z_\beta)\, d\alpha\, d\beta$$

where $y_\alpha = \partial y/\partial\alpha$ (and similarly for $z_\alpha, y_\beta, z_\beta$). Hence, we have two Euler-Lagrange equations:

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{\partial}{\partial\alpha}\left(\frac{\partial F}{\partial y_\alpha}\right) - \frac{\partial}{\partial\beta}\left(\frac{\partial F}{\partial y_\beta}\right) &= 0 \\ \frac{\partial F}{\partial z} - \frac{\partial}{\partial\alpha}\left(\frac{\partial F}{\partial z_\alpha}\right) - \frac{\partial}{\partial\beta}\left(\frac{\partial F}{\partial z_\beta}\right) &= 0 \end{aligned}$$

To generalize this formulation, let denote the independent variables with x_i ($i = 1, \cdots, m$) and the dependent variables with y_j ($j = 1, \cdots, n$). On combining the formulations of § 1.6 and § 1.7 (using this notation), the functional integral is:

$$I[y_1, \cdots, y_n] = \int\!\cdots\!\int_\Omega F(x_1, \cdots, x_m, y_1, \cdots, y_n, y_{11}, \cdots, y_{1m}, \cdots, y_{n1}, \cdots, y_{nm})\, dx_1 \cdots dx_m$$

where $y_{nm} = \partial y_n/\partial x_m$ (and similarly for similar notations). Hence, we have n Euler-Lagrange equations:

$$\frac{\partial F}{\partial y_j} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_{ji}}\right) = 0 \qquad (j = 1, \cdots, n)$$

where $y_{ji} = \partial y_j/\partial x_i$.

2. State the variational formulation for a variational problem with two independent variables (t and x) and one dependent variable y where the functional I depends also on the second order derivatives of the dependent variable y (i.e. $\partial^2 y/\partial t^2$, $\partial^2 y/\partial x\partial t$ and $\partial^2 y/\partial x^2$).
Answer: Combining the formulations of § 1.5 and § 1.6, the functional integral is:

$$I[y] = \iint_\Omega F(t, x, y, y_t, y_x, y_{tt}, y_{tx}, y_{xx})\, dt\, dx$$

---

[38] In fact, this formulation has some restrictions which are not given here (because the example is for demonstration only). For instance, in some problems the multiplier λ (or multipliers λ's) may enter as a dependent variable.

while the Euler-Lagrange equation is:[39]

$$\frac{\partial F}{\partial y}-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right)+\frac{\partial^2}{\partial t^2}\left(\frac{\partial F}{\partial y_{tt}}\right)+\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial F}{\partial y_{tx}}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial y_{xx}}\right)=0$$

where $y_t=\partial y/\partial t$, $y_x=\partial y/\partial x$, $y_{tt}=\partial^2 y/\partial t^2$, $y_{tx}=\partial^2 y/\partial x\partial t$ and $y_{xx}=\partial^2 y/\partial x^2$.

3. State the variational formulation for a variational problem with one independent variable x and two dependent variables (y_1 and y_2) where the functional I depends also on the second order derivative of y_1 and the second and third order derivatives of y_2. Generalize this formulation to variational problems with m dependent variables where the i^{th} dependent variable depends on the derivatives up to (and including) n_i^{th} derivatives.
Answer: Combining the formulations of § 1.5 and § 1.7, the functional integral is:

$$I\left[y_1,y_2\right]=\int_{x_1}^{x_2}F\left(x,y_1,y_2,y_1',y_1'',y_2',y_2'',y_2'''\right)\,dx$$

Hence, we have two Euler-Lagrange equations:

$$\begin{aligned}\frac{\partial F}{\partial y_1}-\frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y_1''}\right) &= 0\\ \frac{\partial F}{\partial y_2}-\frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y_2''}\right)-\frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y_2'''}\right) &= 0\end{aligned}$$

To generalize this formulation, let denote the dependent variables with y_i ($i=1,\cdots,m$) and the highest derivative of y_i with $y_i^{(n_i)}$. On combining the formulations of § 1.5 and § 1.7 (using this notation), the functional integral is:

$$I\left[y_1,\cdots,y_m\right]=\int_{x_1}^{x_2}F\left(x,y_1,y_1',\cdots,y_1^{(n_1)},\cdots,y_m,y_m',\cdots,y_m^{(n_m)}\right)\,dx$$

Hence, we have m Euler-Lagrange equations:

$$\frac{\partial F}{\partial y_i}+\sum_{j=1}^{n_i}(-1)^j\frac{d^j}{dx^j}\left(\frac{\partial F}{\partial y_i^{(j)}}\right)=0 \qquad (i=1,\cdots,m)$$

Note: the highest order derivative for the dependent variables y_i's may differ (as the notation n_i indicates) and hence the highest order derivative of y_1 which I depends on may be the third while the highest order derivative of y_2 which I depends on may be the second. We should also note that some of the derivatives between the first order and the highest order may be missing[40] (e.g. I may depend on the first and third order derivatives of y_1 but not on the second order derivative) and hence the above formulation could be amended accordingly.

4. Extend the generalized formulation of Problem 3 for the case in which the variational problem also includes k constraints $\int G_1,\cdots,\int G_k$.
Answer: We have:

$$I\left[y_1,\cdots,y_m\right]=\int_{x_1}^{x_2}H\left(x,y_1,y_1',\cdots,y_1^{(n_1)},\cdots,y_m,y_m',\cdots,y_m^{(n_m)}\right)\,dx$$

where:

$$H=F+\sum_{i=1}^{k}\lambda_i G_i$$

---

[39] The reader should be careful in interpreting the partial derivatives with respect to the independent variables (as explained in § 1.6).

[40] In fact, in some cases even the first order derivative (and possibly the derivatives of all orders) of some dependent variables can be missing.

Hence, we have m Euler-Lagrange equations:[41]

$$\frac{\partial H}{\partial y_i} + \sum_{j=1}^{n_i} (-1)^j \frac{d^j}{dx^j} \left(\frac{\partial H}{\partial y_i^{(j)}} \right) = 0 \qquad (i = 1, \cdots, m)$$

1.11 Summary

We can summarize the results of the previous sections in Table 1.

Table 1: Summary of the results of sections 1.4-1.10.

Variational Problem	Euler-Lagrange Equation
$F(x, y, y')$	$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ or $\frac{\partial F}{\partial x} - \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$ or $\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y'} - y'\frac{\partial^2 F}{\partial y \partial y'} - y''\frac{\partial^2 F}{\partial y'^2} = 0$
$F(y, y')$	$F - y'\frac{\partial F}{\partial y'} = C$
$F(x, y')$	$\frac{\partial F}{\partial y'} = C$
$F(x, y)$	$\frac{\partial F}{\partial y} = 0$
$F\left(x, y, y^{(1)}, \cdots, y^{(n)}\right)$	$\frac{\partial F}{\partial y} + \sum_{i=1}^{n} (-1)^i \frac{d^i}{dx^i}\left(\frac{\partial F}{\partial y^{(i)}}\right) = 0$
$F(x_1, \cdots, x_n, y, y_{x_1}, \cdots, y_{x_n})$	$\frac{\partial F}{\partial y} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_{x_i}}\right) = 0$
$F(x, y_1, \cdots, y_n, y'_1, \cdots, y'_n)$	$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'_i}\right) = 0 \qquad (i = 1, \cdots, n)$
$F(x, y, y')$ with n constraints $\int G_1, \cdots, \int G_n$	$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$ where $H = F + \sum_{i=1}^{n} \lambda_i G_i$
$F(x, y, y')$ with variable boundary(s)	$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ with transversality condition(s) $F\,dX + (dY - y'dX)\,F_{y'} = 0$
Mixed nature	Mixed formulation (as above)

---

[41] In some cases, it may be required to treat λ's as variational dependent variables and hence other equation(s) may be added (as required). In fact, extra conditions and restrictions are required to make the formulation more rigorous (so the above formulation is mainly for the purpose of demonstrating the idea of mixed techniques in treating the variational problems of mixed nature).

Chapter 2
Optimal Curves

In this chapter we present and solve problems about topics and applications of the mathematics of variation related to optimal curves, i.e. we are looking in these problems to certain curves (or 1D objects) that optimize something (such as length). In fact, the given problems represent just a sample of the variational problems in this category to outline the methods of tackling this sort of problems. There are many other problems of this type that can be considered and solved similarly. This equally applies to the problems of other forthcoming chapters.

2.1 Geodesic Curves

The objective in these problems is to find a curve of optimal length (usually shortest) that connects two boundaries (whether fixed points or variable curves) in a given multi-dimensional space (such as 2D surface or 3D Euclidean space).

Problems

1. Find the equation of geodesic (i.e. curve of shortest length connecting two points) on a Euclidean plane.
   Answer: The length s is the integral of the line element ds along the path Γ that connects the two points, that is $s = \int_\Gamma ds$. In a plane coordinated by an orthonormal Cartesian coordinate system, the line element ds is given by $ds = \sqrt{(dx)^2 + (dy)^2}$ (see Figure 11) and hence the length (which represents the functional I that we are supposed to optimize) is given by:

$$s = \int_\Gamma \sqrt{(dx)^2 + (dy)^2} = \int_{x_A}^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_A}^{x_B} \sqrt{1 + y'^2}\, dx \equiv I\left[y\right]$$

   where the prime means d/dx.[42] On comparing this to Eq. 1 we can see that $F\left(x, y, y'\right) \equiv \sqrt{1 + y'^2}$. If we now apply the Euler-Lagrange equation (i.e. Eq. 2) we get:[43]

$$\begin{aligned}
\frac{\partial\sqrt{1+y'^2}}{\partial y} - \frac{d}{dx}\left(\frac{\partial\sqrt{1+y'^2}}{\partial y'}\right) &= 0 \\
0 - \frac{d}{dx}\left(\frac{2y'}{2\sqrt{1+y'^2}}\right) &= 0 \\
\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) &= 0 \\
\frac{y'}{\sqrt{1+y'^2}} &= D \qquad (D \text{ is constant})
\end{aligned}$$

---

[42] We note that the line element ds (in Cartesian coordinates) may be cast in another form, that is:

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy = \sqrt{1 + x'^2}\, dy$$

where the prime here means d/dy. This form may be used in solving some problems and may have certain advantages (such as simplifying the solution) over the common form in some cases.

[43] Because F depends on y' only, we can also use Eq. 4 or footnote [12]. However, we use Eq. 2 for more practice.

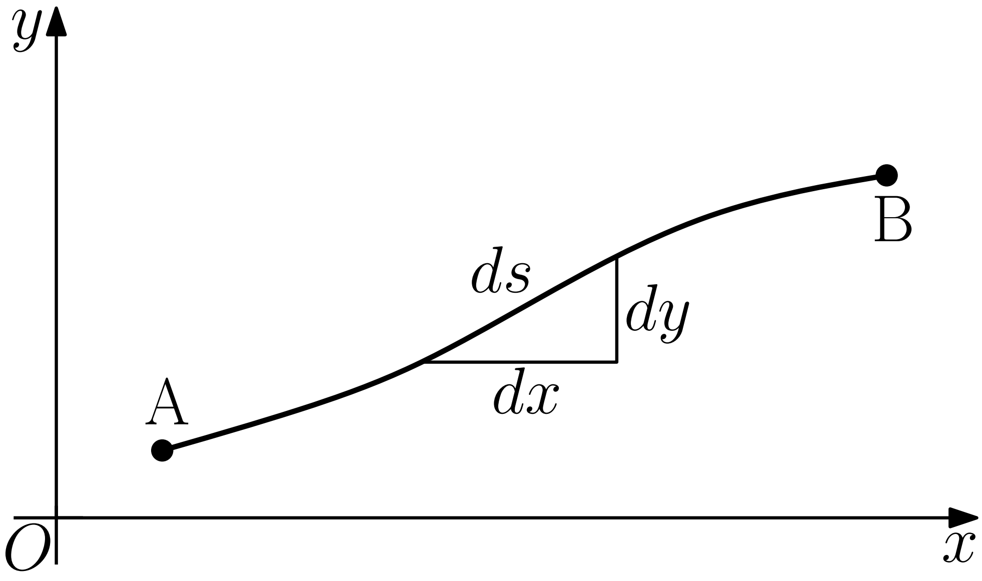

Figure 11: A simple sketch depicting the setting of the Problem of shortest distance connecting two points (A and B) on a plane with ds representing infinitesimal arc length (or line element). See Problem 1 of § 2.1.

$$\begin{aligned} y'^2 &= D^2\left(1+y'^2\right) \\ y'^2\left(1-D^2\right) &= D^2 \\ y' &= a && \left(a=\pm\sqrt{D^2/\left(1-D^2\right)} \text{ is constant}\right) \\ y &= ax+b && (b \text{ is constant}) \end{aligned}$$

which is an equation of a straight line. So, the shortest distance between two points on a plane is the length of the straight line segment that connects these two points, (i.e. on a Euclidean plane the geodesic is a straight line).
Note: it is obvious that the optimal length in this Problem is a minimum (not a maximum) because the length of a curve connecting two points can diverge.

2. Re-solve Problem 1 using this time plane polar coordinates (ρ, ϕ).
Answer: In polar coordinates $ds = \sqrt{(d\rho)^2 + \rho^2 (d\phi)^2}$ and hence:

$$s = \int_\Gamma \sqrt{(d\rho)^2 + \rho^2 (d\phi)^2} = \int_{\rho_A}^{\rho_B} \sqrt{1+\rho^2 (d\phi/d\rho)^2}\, d\rho = \int_{\rho_A}^{\rho_B} \sqrt{1+\rho^2\phi'^2}\, d\rho \equiv I[\phi]$$

where the prime means $d/d\rho$. On comparing this to Eq. 1 (noting that x, y, y' in Eq. 1 correspond to ρ, ϕ, ϕ' here) we can see that $F(\rho, \phi, \phi') \equiv \sqrt{1+\rho^2\phi'^2}$. If we now apply the Euler-Lagrange equation (i.e. Eq. 4 noting that F is independent of ϕ which corresponds to y) we get:

$$\begin{aligned} \frac{\partial\sqrt{1+\rho^2\phi'^2}}{\partial\phi'} &= C \\ \frac{\rho^2\phi'}{\sqrt{1+\rho^2\phi'^2}} &= C \\ \rho^4\phi'^2 &= C^2\left(1+\rho^2\phi'^2\right) \\ \phi'^2 &= \frac{C^2}{\rho^4 - C^2\rho^2} \end{aligned}$$

$$\begin{aligned}
\phi' &= \pm\frac{C}{\sqrt{\rho^4 - C^2\rho^2}} \\
\phi + \phi_0 &= \pm\arctan\left(\frac{\sqrt{\rho^2 - C^2}}{C}\right) \\
\pm\tan(\phi + \phi_0) &= \frac{\sqrt{\rho^2 - C^2}}{C} \\
C^2\tan^2(\phi + \phi_0) &= \rho^2 - C^2 \\
C^2\left[\tan^2(\phi + \phi_0) + 1\right] &= \rho^2 \\
C^2\sec^2(\phi + \phi_0) &= \rho^2 \\
\rho^2\cos^2(\phi + \phi_0) &= C^2 \\
\rho\cos(\phi + \phi_0) &= \pm C
\end{aligned}$$

This is an equation of a straight line as can be shown as follows:

$$\begin{aligned}
\rho\cos(\phi + \phi_0) &= \pm C \\
\rho(\cos\phi\cos\phi_0 - \sin\phi\sin\phi_0) &= \pm C \qquad \text{(trigonometric identity)} \\
\rho\cos\phi\cos\phi_0 - \rho\sin\phi\sin\phi_0 &= \pm C \\
x\cos\phi_0 - y\sin\phi_0 &= \pm C \qquad \text{(transforming to Cartesian)}
\end{aligned}$$

As we see, this is an equation of a straight line (noting that ϕ_0 and C are constants).
Note: when we take the square root in Problems like this we usually use ± for the sake of formality although C (and its alike in similar Problems) is usually arbitrary and hence it can represent both signs.

3. Find the equation of geodesic in 3D Euclidean space.
Answer: This Problem is similar to Problem 1. We use Cartesian coordinates x, y, z and hence $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{1 + y'^2 + z'^2}\,dx$ (where the prime means d/dx). Hence, the distance (which represents the functional I that we are supposed to optimize) is given by:

$$s = \int_\Gamma ds = \int_{x_A}^{x_B} \sqrt{1 + y'^2 + z'^2}\,dx \equiv I[y, z]$$

So, $F(x, y, z, y', z') \equiv \sqrt{1 + y'^2 + z'^2}$ (noting that this is a Problem with multiple dependent variables; see § 1.7). Accordingly, the Euler-Lagrange equations for the dependent variables y and z are given by Eqs. 19 and 20 (noting that y_1, y_2 there correspond to y, z here).
The y equation is:

$$\begin{aligned}
\frac{\partial\sqrt{1 + y'^2 + z'^2}}{\partial y} - \frac{d}{dx}\left(\frac{\partial\sqrt{1 + y'^2 + z'^2}}{\partial y'}\right) &= 0 \\
0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) &= 0 \\
\frac{y'}{\sqrt{1 + y'^2 + z'^2}} &= C \\
y'^2 &= C^2\left(1 + y'^2 + z'^2\right) \\
y'^2\left(1 - C^2\right) &= C^2\left(1 + z'^2\right) \qquad (30)
\end{aligned}$$

The z equation is:

$$\frac{\partial\sqrt{1 + y'^2 + z'^2}}{\partial z} - \frac{d}{dx}\left(\frac{\partial\sqrt{1 + y'^2 + z'^2}}{\partial z'}\right) = 0$$

$$\begin{aligned}
0-\frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) &= 0 \\
\frac{z'}{\sqrt{1+y'^2+z'^2}} &= D \\
z'^2 &= D^2\left(1+y'^2+z'^2\right) \\
z'^2\left(1-D^2\right) &= D^2\left(1+y'^2\right) \qquad (31)
\end{aligned}$$

On substituting from Eq. 31 into Eq. 30 (see the upcoming note 1) and simplifying we get $y' =$ constant and hence $y = ax + b$ (with a and b being constants) which is an equation of a plane. Similarly, on substituting from Eq. 30 into Eq. 31 (see the upcoming note 1) and simplifying we get $z' =$ constant and hence $z = cx + d$ (with c and d being constants) which is also an equation of a plane. So, the solution of the system of the y and z equations is the intersection of these planes which is a straight line. Accordingly, the geodesic (or the curve of shortest length) in this case is a straight line.

Note 1: from Eq. 31 we get:

$$z'^2=\frac{D^2\left(1+y'^2\right)}{1-D^2}$$

On substituting from this equation into Eq. 30 we get:

$$\begin{aligned}
y'^2\left(1-C^2\right) &= C^2\left(1+\frac{D^2\left(1+y'^2\right)}{1-D^2}\right) \\
y'^2\left(1-C^2\right) &= C^2+\frac{C^2D^2}{1-D^2}+\frac{C^2D^2y'^2}{1-D^2} \\
y'^2\left(1-C^2-\frac{C^2D^2}{1-D^2}\right) &= C^2+\frac{C^2D^2}{1-D^2} \\
y'^2 &= \left(C^2+\frac{C^2D^2}{1-D^2}\right)\left(1-C^2-\frac{C^2D^2}{1-D^2}\right)^{-1} \\
y' &= \text{constant}
\end{aligned}$$

We similarly obtain $z' =$ constant (with the exchange of y' and z' and C and D).

Note 2: it is obvious that the optimal length in this Problem is a minimum (not a maximum) because the length of a curve connecting two points can diverge.

4. Find the equation of geodesic on a right circular cylinder.

Answer: We use cylindrical coordinates ρ, ϕ, z to represent the cylinder where the axis of the cylinder coincides with the z axis of the coordinate system. Now, the line element ds in cylindrical coordinates is given by:

$$ds=\sqrt{\left(d\rho\right)^2+\rho^2\left(d\phi\right)^2+\left(dz\right)^2}$$

For right circular cylinder ρ is constant and hence $d\rho = 0$ and $\rho = R$ (with R being the constant radius of the cylinder). Therefore, the line element of the cylinder is:

$$ds=\sqrt{R^2\left(d\phi\right)^2+\left(dz\right)^2}=\sqrt{R^2+\left(dz/d\phi\right)^2}\,d\phi=\sqrt{R^2+z'^2}\,d\phi$$

where $z' = dz/d\phi$. Hence, the length of the curve Γ (which is the functional I that we intend to optimize) is:

$$s=\int_\Gamma ds=\int_{\phi_1}^{\phi_2}\sqrt{R^2+z'^2}\,d\phi\equiv I\left[z\right]$$

where ϕ_1, ϕ_2 represent the azimuthal coordinates of the end points of the curve. On comparing this to Eq. 1 (noting that x, y, y' in Eq. 1 correspond to ϕ, z, z' here) we can see that $F \equiv \sqrt{R^2 + z'^2}$.

Considering that F has no explicit dependency on ϕ (which corresponds to x in our case), we can use the Beltrami identity (i.e. Eq. 3 with the replacement of y' with z'), that is:[44]

$$\begin{aligned}
\sqrt{R^2+z'^2}-z'\frac{\partial\sqrt{R^2+z'^2}}{\partial z'} &= C\\
\sqrt{R^2+z'^2}-z'\left(\frac{2z'}{2\sqrt{R^2+z'^2}}\right) &= C\\
\sqrt{R^2+z'^2}-\frac{z'^2}{\sqrt{R^2+z'^2}} &= C\\
R^2+z'^2-z'^2 &= C\sqrt{R^2+z'^2}\\
R^4 &= C^2\left(R^2+z'^2\right)\\
z'^2 &= \frac{R^4}{C^2}-R^2\\
z' &= \pm\sqrt{\frac{R^4}{C^2}-R^2}\\
\frac{dz}{d\phi} &= \pm D \qquad \left(D=\sqrt{\frac{R^4}{C^2}-R^2}\right)\\
z &= \pm D\phi+E \qquad (E\text{ is constant}) \qquad (32)
\end{aligned}$$

which is an equation of a circular helix.

Note: the helix can represent a circular arc as a special case when $D = 0$ (in the case where the two points connected by the geodesic are on a circle). It can also represent a generator line parallel to the z axis when $D \to \infty$ (in the case where the two points connected by the geodesic are on a generator).

5. Re-solve Problem 4 using this time a multiple dependent variables approach (see § 1.7).

Answer: We use t-parameterized cylindrical coordinates $\rho(t), \phi(t), z(t)$ to represent the cylinder where the axis of the cylinder coincides with the z axis of the coordinate system. Now, the line element ds in cylindrical coordinates is given by:

$$ds=\sqrt{(d\rho)^2+\rho^2(d\phi)^2+(dz)^2}$$

For right circular cylinder ρ is constant and hence $d\rho = 0$ and $\rho = R$ (with R being the constant radius of the cylinder). Therefore, the line element of the cylinder is:

$$ds=\sqrt{R^2(d\phi)^2+(dz)^2}=\sqrt{R^2(d\phi/dt)^2+(dz/dt)^2}\,dt=\sqrt{R^2\dot\phi^2+\dot z^2}\,dt$$

where the overdot represents d/dt. Hence, the length of the curve Γ (which is the functional I that we intend to optimize) is:

$$s=\int_\Gamma ds=\int_{t_1}^{t_2}\sqrt{R^2\dot\phi^2+\dot z^2}\,dt\equiv I[\phi,z]$$

where t_1, t_2 represent the values of the parameter t at the end points of the curve. On comparing this equation to Eq. 18 (noting that x, y_1, y_2, y'_1, y'_2 in Eq. 18 correspond to $t, \phi, z, \dot\phi, \dot z$ here) we can see that $F \equiv \sqrt{R^2\dot\phi^2+\dot z^2}$. Accordingly, the Euler-Lagrange equations for the dependent variables ϕ and z are given by Eqs. 19 and 20 (with the replacement of x, y_1, y_2, y'_1, y'_2 with $t, \phi, z, \dot\phi, \dot z$).

The ϕ equation is:

$$\frac{\partial\sqrt{R^2\dot\phi^2+\dot z^2}}{\partial\phi}-\frac{d}{dt}\left(\frac{\partial\sqrt{R^2\dot\phi^2+\dot z^2}}{\partial\dot\phi}\right) = 0$$

---

[44] Noting that only z' appears in F we can use footnote [12] (or Eq. 4) to conclude z' = constant immediately. However, we use here the Beltrami identity for diversity and practice (as well as showing that these equations lead to the same result).

$$
\begin{aligned}
0-\frac{d}{dt}\left(\frac{2R^2\dot{\phi}}{2\sqrt{R^2\dot{\phi}^2+\dot{z}^2}}\right) &= 0 \\
\frac{R^2\dot{\phi}}{\sqrt{R^2\dot{\phi}^2+\dot{z}^2}} &= C_1 \\
\dot{\phi}^2 &= \frac{C_1^2}{R^4}\left(R^2\dot{\phi}^2+\dot{z}^2\right) \\
\dot{z}^2 &= \left(\frac{R^4}{C_1^2}-R^2\right)\dot{\phi}^2 \\
\dot{z} &= \pm D_1\dot{\phi} \qquad\qquad \left(D_1=\sqrt{\frac{R^4}{C_1^2}-R^2}\right) \\
z &= \pm D_1\phi+E_1 \qquad\qquad (E_1 \text{ is constant})
\end{aligned}
$$

which is an equation of a circular helix.
Similarly, the z equation is:

$$
\begin{aligned}
\frac{\partial\sqrt{R^2\dot{\phi}^2+\dot{z}^2}}{\partial z}-\frac{d}{dt}\left(\frac{\partial\sqrt{R^2\dot{\phi}^2+\dot{z}^2}}{\partial\dot{z}}\right) &= 0 \\
0-\frac{d}{dt}\left(\frac{2\dot{z}}{2\sqrt{R^2\dot{\phi}^2+\dot{z}^2}}\right) &= 0 \\
\frac{\dot{z}}{\sqrt{R^2\dot{\phi}^2+\dot{z}^2}} &= C_2 \\
\dot{z}^2 &= C_2^2R^2\dot{\phi}^2+C_2^2\dot{z}^2 \\
\dot{z}^2 &= \left(\frac{C_2^2R^2}{1-C_2^2}\right)\dot{\phi}^2 \\
\dot{z} &= \pm D_2\dot{\phi} \qquad\qquad \left(D_2=\sqrt{\frac{C_2^2R^2}{1-C_2^2}}\right) \\
z &= \pm D_2\phi+E_2 \qquad\qquad (E_2 \text{ is constant})
\end{aligned}
$$

which is also an equation of a circular helix.
So, the solution from both equations is the same (which is also the same as the solution obtained in Problem 4; see Eq. 32). As before (see the note of Problem 4), the helix can also represent a circular arc (i.e. an arc of a circle $r = R$) as a special case. It can also represent a generator (i.e. line parallel to the z axis) as another special case.
Note 1: because a circular helix (on a right circular cylinder) passing through two given points is unique (within certain conditions; see note 2), all the above equations (i.e. $z = \pm D\phi + E$ of Problem 4 and $z = \pm D_1\phi + E_1$ and $z = \pm D_2\phi + E_2$ of the present Problem) represent the same helix.
Note 2: we are assuming a given handedness (as well as being optimal in a specific sense). Anyway, being unique or non-unique is related to optimizing in a global sense or stationarizing (noting that in this regard there are some details that can be worked out rather easily). In brief, we can say more explicitly: although there are generally infinitely many circular helices that can pass through two given points on a cylinder, only one of these is optimal in the specific (global) geodesic sense with a given handedness. In fact, the uniqueness of geodesic (in a global sense) should be achieved if we restrict the magnitude of the change in ϕ for the geodesic to the range $0 \le |\Delta\phi| \le \pi$ (where $\Delta\phi = \phi_2 - \phi_1$ with ϕ_1 and ϕ_2 being the azimuthal coordinates of the end points) with a conventional selection of a

specific handedness for the case $|\Delta\phi| = \pi$. We should finally note that part of the confusion about these issues may originate from the meaning of ϕ and if it belongs to the coordinate system and hence to the cylinder (where $0 \le \phi < 2\pi$) or it belongs to the helix that connects the two end points (and hence ϕ represents the "spin" of the helix regardless of the aforementioned restriction on ϕ). In other words, whether the helix (as a function of ϕ) is represented as one-to-many or as one-to-one. Also see Problem 6.

6. A right circular cylinder has a radius $R = 6$ with its axis being aligned along the z axis of a cylindrical coordinate system. Find the equation of the geodesic on this cylinder that passes through the point $(\phi_A, z_A) = (\frac{\pi}{2}, 1)$ and the point $(\phi_B, z_B) = (\pi, 6)$.
   Answer: Inserting the coordinates of these points into Eq. 32 (noting that the sign is rather arbitrary), we get:

$$\begin{aligned} 1 &= D\frac{\pi}{2} + E \\ 6 &= D\pi + E \end{aligned}$$

   On solving these equations we get $D = \frac{10}{\pi}$ and $E = -4$. Accordingly, the geodesic on this cylinder that passes through those points is a helix (or rather helical arc) described by the equations:

$$z = \frac{10}{\pi}\phi - 4 \qquad \text{and} \qquad \rho = 6$$

   Note: if we used the minus sign in Eq. 32 we get $D = -\frac{10}{\pi}$ and $E = -4$ and hence $z = \frac{10}{\pi}\phi - 4$ which is the same.

7. Find the equation of geodesic on a sphere.
   Answer: We use a spherical coordinate system r, θ, ϕ centered on the center of the sphere. Now, the line element ds in spherical coordinates is given by:

$$ds = \sqrt{(dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta\,(d\phi)^2}$$

   For sphere, r is constant and hence $dr = 0$ and $r = R$ (with R being the constant radius of the sphere). Therefore, the line element on a sphere is:

$$ds = \sqrt{R^2 (d\theta)^2 + R^2 \sin^2\theta\,(d\phi)^2} = R\sqrt{(d\theta)^2 + \sin^2\theta\,(d\phi)^2} = R\sqrt{1 + \phi'^2 \sin^2\theta}\,d\theta$$

   where $\phi' = d\phi/d\theta$. Hence, the length of the curve Γ (which is the functional I that we intend to optimize) is:

$$s = \int_\Gamma ds = R\int_{\theta_1}^{\theta_2} \sqrt{1 + \phi'^2 \sin^2\theta}\,d\theta \equiv I[\phi]$$

   where θ_1, θ_2 represent the θ coordinates of the end points of the curve. On comparing this to Eq. 1 (noting that x, y, y' in Eq. 1 correspond to θ, ϕ, ϕ' here) we can see that $F \equiv \sqrt{1 + \phi'^2 \sin^2\theta}$ (noting that R is a constant and hence it is no more than a scaling factor). Considering that F has no explicit dependency on ϕ (which corresponds to y in our case), we can use Eq. 4 (with the replacement of y' with ϕ'), that is:

$$\begin{aligned} \frac{\partial}{\partial\phi'}\left(\sqrt{1 + \phi'^2 \sin^2\theta}\right) &= C \\ \frac{2\phi' \sin^2\theta}{2\sqrt{1 + \phi'^2 \sin^2\theta}} &= C \\ \phi'^2 \sin^4\theta &= C^2 + C^2\phi'^2 \sin^2\theta \\ \phi'^2 &= \frac{C^2}{\sin^4\theta - C^2 \sin^2\theta} \end{aligned}$$

$$\begin{aligned}
\phi'^2 &= \frac{C^2}{\sin^4\theta\left(1 - C^2\sin^{-2}\theta\right)} \\
\phi'^2 &= \frac{C^2\csc^4\theta}{(1 - C^2\csc^2\theta)} \\
\phi' &= \frac{C\csc^2\theta}{\sqrt{1 - C^2 - C^2\cot^2\theta}} && (\csc^2\theta = 1 + \cot^2\theta) \\
\phi' &= \frac{\csc^2\theta}{\sqrt{\left(\frac{1-C^2}{C^2}\right) - \cot^2\theta}} \\
d\phi &= \frac{\csc^2\theta\, d\theta}{\sqrt{\left(\frac{1-C^2}{C^2}\right) - \cot^2\theta}}
\end{aligned}$$

Now, let $D^2 = C^2/(1 - C^2)$ and $w = \cot\theta$ (and hence $dw = -\csc^2\theta\, d\theta$). On substituting these in the last equation and integrating we get:

$$\begin{aligned}
\int d\phi &= -\int \frac{dw}{\sqrt{\frac{1}{D^2} - w^2}} \\
\phi &= -\arcsin(Dw) + \phi_0 && (\phi_0 \text{ is constant}) \\
\phi_0 - \phi &= \arcsin(D\cot\theta) \\
\sin(\phi_0 - \phi) &= D\cot\theta \\
\sin\phi_0\cos\phi - \cos\phi_0\sin\phi &= D\frac{\cos\theta}{\sin\theta} && (\text{trigonometric identities}) \\
\sin\phi_0 R\sin\theta\cos\phi - \cos\phi_0 R\sin\theta\sin\phi &= DR\cos\theta && (\text{multiplying with } R\sin\theta) \\
(\sin\phi_0)\, x - (\cos\phi_0)\, y &= Dz
\end{aligned}$$

where in the last line we transformed to Cartesian coordinates. As we see, the last equation is an equation of a plane passing through the origin (which is the center of the sphere) and hence the geodesic curve on a sphere is an arc of a great circle of the sphere (where the great circle is the intersection of the plane and the sphere).

8. Find the equation of geodesic on a right circular cone.
Answer: We use spherical coordinates r, θ, ϕ to represent the cone where the apex of the cone is at the origin of coordinates while the axis of the cone coincides with the $\theta = 0$ axis of the coordinate system. Now, the line element ds in spherical coordinates is given by:

$$ds = \sqrt{(dr)^2 + r^2\,(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2}$$

For right circular cone (according to the above setting), θ is constant and hence $d\theta = 0$ and $\theta = \alpha$ (with α being a constant angle). Therefore, the line element on a cone is:

$$ds = \sqrt{(dr)^2 + r^2\sin^2\alpha\,(d\phi)^2} = \sqrt{(dr/d\phi)^2 + r^2\sin^2\alpha}\; d\phi = \sqrt{r'^2 + r^2\sin^2\alpha}\; d\phi$$

where $r' = dr/d\phi$. Hence, the length of the curve Γ (which is the functional I that we intend to optimize) is:

$$s = \int_\Gamma ds = \int_{\phi_1}^{\phi_2} \sqrt{r'^2 + r^2\sin^2\alpha}\, d\phi \equiv I\,[r]$$

where ϕ_1, ϕ_2 represent the azimuthal coordinates of the end points of the curve. On comparing this to Eq. 1 (noting that x, y, y' in Eq. 1 correspond to ϕ, r, r' here) we can see that $F \equiv \sqrt{r'^2 + r^2\sin^2\alpha}$.

Considering that F has no explicit dependency on ϕ (which corresponds to x in our case), we can use the Beltrami identity (i.e. Eq. 3 noting that y' in Eq. 3 correspond to r' here), that is:

$$\begin{aligned}
\sqrt{r'^2 + r^2 \sin^2 \alpha} - r' \frac{\partial}{\partial r'} \left(\sqrt{r'^2 + r^2 \sin^2 \alpha} \right) &= C \\
\sqrt{r'^2 + r^2 \sin^2 \alpha} - r' \left(\frac{2r'}{2\sqrt{r'^2 + r^2 \sin^2 \alpha}} \right) &= C \\
\sqrt{r'^2 + r^2 \sin^2 \alpha} - \frac{r'^2}{\sqrt{r'^2 + r^2 \sin^2 \alpha}} &= C \\
r'^2 + r^2 \sin^2 \alpha - r'^2 &= C\sqrt{r'^2 + r^2 \sin^2 \alpha} \\
r^2 \sin^2 \alpha &= C\sqrt{r'^2 + r^2 \sin^2 \alpha} \\
r^4 \sin^4 \alpha &= C^2 \left(r'^2 + r^2 \sin^2 \alpha \right) \\
r'^2 &= \frac{r^4 \sin^4 \alpha}{C^2} - r^2 \sin^2 \alpha \\
r'^2 &= ar^4 - br^2 \qquad (a \text{ and } b \text{ are constants}) \\
r' &= \sqrt{ar^4 - br^2} \\
\frac{dr}{d\phi} &= \sqrt{ar^4 - br^2} \\
d\phi &= \frac{dr}{\sqrt{ar^4 - br^2}} \\
\phi &= \frac{1}{\sqrt{b}} \arctan \left(\sqrt{\frac{ar^2 - b}{b}} \right) + D \qquad (33)
\end{aligned}$$

where the result of the last line can be easily checked by differentiation (to obtain the results of the earlier lines).

9. Re-solve Problem 8 using this time cylindrical coordinates (ρ, ϕ, z).
Answer: Let the axis of the cone coincide with the (positive) z axis of the coordinate system while the apex of the cone be at the origin of coordinates. Now, the line element ds in cylindrical coordinates is given by:

$$ds = \sqrt{(d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2}$$

For right circular cone z is proportional to ρ and hence $z = k\rho$ where k is a constant. Therefore, the line element of the cone becomes:

$$ds = \sqrt{(d\rho)^2 + \rho^2 (d\phi)^2 + (kd\rho)^2} = \sqrt{1 + k^2 + \rho^2 \phi'^2}\, d\rho$$

where $\phi' = d\phi/d\rho$. Hence, the length of the curve Γ (which is the functional I that we intend to optimize) is:

$$s = \int_\Gamma ds = \int_{\rho_1}^{\rho_2} \sqrt{1 + k^2 + \rho^2 \phi'^2}\, d\rho \equiv I[\phi]$$

where ρ_1, ρ_2 represent the ρ coordinates of the end points of the curve. On comparing this to Eq. 1 (noting that x, y, y' in Eq. 1 correspond to ρ, ϕ, ϕ' here) we can see that $F \equiv \sqrt{1 + k^2 + \rho^2 \phi'^2}$. Considering that F has no explicit dependency on ϕ (which corresponds to y in our case), we can use Eq. 4 (with the replacement of y' with ϕ'), that is:

$$\begin{aligned}
\frac{\partial F}{\partial \phi'} &= C \\
\frac{\rho^2 \phi'}{\sqrt{1 + k^2 + \rho^2 \phi'^2}} &= C
\end{aligned}$$

$$\begin{aligned}
\rho^4\phi'^2 &= C^2 + C^2k^2 + C^2\rho^2\phi'^2 \\
\phi'^2 &= \frac{C^2 + C^2k^2}{\rho^4 - C^2\rho^2} \\
\frac{d\phi}{d\rho} &= \sqrt{\frac{C^2 + C^2k^2}{\rho^4 - C^2\rho^2}} \\
\phi &= \sqrt{1+k^2}\arctan\left(\frac{\sqrt{\rho^2 - C^2}}{C}\right) + D \qquad (34)
\end{aligned}$$

Note: the similarity in form between Eq. 33 and Eq. 34 is because on a right circular cone (coordinated as described in Problem 8 and in the present Problem) ρ of the cylindrical coordinates is proportional to r of the spherical coordinates (since $\rho = r\sin\alpha$) noting that ϕ is the same in both coordinate systems (assuming that the systems are in a standard configuration with respect to a corresponding orthonormal Cartesian system as we do).

10. Find the equation of geodesic on a surface of revolution.
Answer: Let use cylindrical coordinates ρ, ϕ, z where the z axis of the coordinate system is the axis of the surface of revolution and hence the profile curve (i.e. meridian) of the surface of revolution is given as $\rho = f(z)$. Now, the line element ds in cylindrical coordinates is given by:

$$ds = \sqrt{(d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2} = \sqrt{\rho'^2 + \rho^2\phi'^2 + 1}\,dz = \sqrt{f'^2 + f^2\phi'^2 + 1}\,dz$$

where the prime means d/dz. Hence, the length of the curve Γ (which is the functional I that we intend to optimize) is:

$$s = \int_\Gamma ds = \int_{z_1}^{z_2} \sqrt{f'^2 + f^2\phi'^2 + 1}\,dz \equiv I[\phi]$$

where z_1, z_2 represent the z coordinates of the end points of the curve. On comparing this to Eq. 1 (noting that x, y, y' in Eq. 1 correspond to z, ϕ, ϕ' here) we can see that $F \equiv \sqrt{f'^2 + f^2\phi'^2 + 1}$. Considering that F has no explicit dependency on ϕ (which corresponds to y in our case), we can use Eq. 4 (with the replacement of y' with ϕ'), that is:

$$\begin{aligned}
\frac{\partial\sqrt{f'^2 + f^2\phi'^2 + 1}}{\partial\phi'} &= C \\
\frac{f^2\phi'}{\sqrt{f'^2 + f^2\phi'^2 + 1}} &= C \qquad (35) \\
f^4\phi'^2 &= C^2\left(f'^2 + f^2\phi'^2 + 1\right) \\
\phi'^2 &= \frac{C^2f'^2 + C^2}{f^4 - C^2f^2} \\
\phi' &= \pm C\sqrt{\frac{f'^2 + 1}{f^4 - C^2f^2}} \qquad (36) \\
\phi &= \pm C\int\sqrt{\frac{f'^2 + 1}{f^4 - C^2f^2}}\,dz \qquad (37)
\end{aligned}$$

which is the equation of geodesic on a surface of revolution (in cylindrical coordinates).

11. Apply the result of Problem 10 on the right circular cylinder and hence confirm the result of Problem 4.
Answer: Using the setting of Problem 4 we have $\rho \equiv f(z) = R$ where R is the (constant) radius of the cylinder. So, from Eq. 37 (with $f = R$ and $f' = 0$) we have:

$$\pm\phi = C\int\sqrt{\frac{0^2 + 1}{R^4 - C^2R^2}}\,dz$$

$$\begin{aligned}
\pm\phi &= C\sqrt{\frac{1}{R^4 - C^2R^2}}\,z + C_1 \\
z &= \pm D\phi + E
\end{aligned}$$

where the constants D and E are defined accordingly. The last equation is the same as Eq. 32 of Problem 4.

12. Apply the result of Problem 10 on the right circular cone and hence confirm the result of Problem 9.
Answer: Using the setting of Problem 9 we have $\rho \equiv f(z) = \kappa z$ where $\kappa = 1/k$. So, from Eq. 37 (with $f = \kappa z$ and $f' = \kappa$) we get:

$$\begin{aligned}
\phi &= C\int\sqrt{\frac{\kappa^2+1}{\kappa^4z^4 - C^2\kappa^2z^2}}\,dz \\
&= \frac{C\sqrt{\kappa^2+1}}{\kappa^2}\int\frac{1}{\sqrt{z^4 - (C/\kappa)^2 z^2}}\,dz \\
&= \frac{C\sqrt{\kappa^2+1}}{\kappa^2\,(C/\kappa)}\arctan\left(\frac{\sqrt{z^2-(C/\kappa)^2}}{C/\kappa}\right) + D \\
&= \frac{\sqrt{\kappa^2+1}}{\kappa}\arctan\left(\frac{\sqrt{(\rho/\kappa)^2-(C/\kappa)^2}}{C/\kappa}\right) + D \\
&= \sqrt{1+(1/\kappa^2)}\arctan\left(\frac{\sqrt{\rho^2-C^2}}{C}\right) + D \\
&= \sqrt{1+k^2}\arctan\left(\frac{\sqrt{\rho^2-C^2}}{C}\right) + D
\end{aligned}$$

The last equation is the same as Eq. 34 of Problem 9.

13. Show that meridians of a surface of revolution are geodesic curves.
Answer: Meridians are represented (in the cylindrical coordinates of Problem 10) by the condition $\phi' = 0$ and hence from Eq. 36 or Eq. 35 (with $C = 0$) we get $\phi =$ constant (which is the geodesic equation of meridians). So, meridians of a surface of revolution are geodesic curves. In other words, Eq. 37 (which is the equation of geodesic on a surface of revolution) represents meridians when $C = 0$ (which leads to $\phi =$ constant of integration) and hence meridians are geodesics.

14. Confirm that (arcs of) great circles of spheres are geodesic curves.
Answer: This is a consequence of the result of Problem 13 because (arcs of) great circles of spheres are meridians (noting the spherical symmetry of spheres). This is inline with the result of Problem 7 (noting that the result of the present Problem is weaker than the result of Problem 7 because in Problem 7 we effectively have "curves on spheres are geodesics *iff* they are great circles" while in the present Problem we effectively have "curves on spheres are geodesics *if* they are great circles").

15. Find the shortest distance between the point $(0, 2)$ and the cubic curve $y = x^3$.
Answer: This is obviously a geodesic problem (since it is about shortest distance) on a Euclidean plane with one fixed boundary and one variable boundary. So, the solution is obviously a straight line $y = ax + b$ whose length between the point $(0, 2)$ and the point B (which is on the curve $y = x^3$) is the shortest distance (see Problem 1). Moreover, from § 1.9 we should have a transversality condition as well (see Eq. 23) at point B, that is:[45]

$$F + (Y' - y')\,F_{y'} = 0 \qquad \text{(at point B)}$$

---

[45] In brief, in this Problem (and its alike) the straight line solution is obtained from the result of Problem 1 (because any geodesic on a Euclidean plane should be a straight line) and hence the job of the transversality condition is to identify the exact point (i.e. point B) at which the straight line and the boundary curve (i.e. the curve $y = x^3$ in our case) meet.

$$\begin{aligned}
\sqrt{1+y'^2}+(Y'-y')\frac{y'}{\sqrt{1+y'^2}} &= 0 && (F=\sqrt{1+y'^2};\ \text{see Problem 1})\\
\sqrt{1+a^2}+\left(3x^2-a\right)\frac{a}{\sqrt{1+a^2}} &= 0 && (y=ax+b \text{ and } Y=x^3)\\
1+a^2+3ax^2-a^2 &= 0 && (\times\sqrt{1+a^2})\\
1+3ax_B^2 &= 0 && (38)
\end{aligned}$$

where in the last equation we used x_B to indicate that this equation applies to point B. Now, the geodesic $y = ax + b$ should pass through point $(0, 2)$ and hence $2 = 0 + b$, i.e. $b = 2$. Also, point B is on both the geodesic $y = ax + 2$ and the boundary curve $y = x^3$ and hence:

$$ax_B+2=x_B^3 \qquad \to \qquad a=\frac{x_B^3-2}{x_B} \qquad (x_B\neq 0) \tag{39}$$

On substituting from the last equation into Eq. 38 and simplifying we get $3x_B^4 - 6x_B + 1 = 0$. On solving this quartic equation we get:

$$\begin{aligned}
x_B &\simeq 0.16705608755 && \text{and hence } a\simeq -11.94411932 \quad \text{and} \quad y_B\simeq 0.00466216\\
&\text{OR}\\
x_B &\simeq 1.19858497157 && \text{and hence } a\simeq -0.23202837 \quad \text{and} \quad y_B\simeq 1.72189428
\end{aligned}$$

Now, if we compute (using Pythagoras) the distance d_1 between point $(0, 2)$ and point $(x_B, y_B) = (0.16705, 0.00466)$ and the distance d_2 between point $(0, 2)$ and point $(x_B, y_B) = (1.19858, 1.72189)$ we get $d_1 \simeq 2.00231887$ and $d_2 \simeq 1.23042624$. So, the geodesic (i.e. curve of shortest length) is $y = -0.23202837x + 2$ and the shortest distance is $d_2 \simeq 1.23042624$. For clarity, we plot the result in Figure 12.

Note: in Eq. 39 we excluded $x_B = 0$ so we need to test the possibility that the geodesic is the line segment connecting point $(0, 2)$ to point $(0, 0)$ on the curve $y = x^3$. However, this line segment cannot be the required geodesic because its length is 2 which is longer than d_2.

16. Find the shortest distance between the parabola $y = x^2$ and the straight line $y = x - 5$.
Answer: This is obviously a geodesic problem (since it is about shortest distance) on a Euclidean plane with two variable boundaries. So, the solution is obviously a straight line $y = ax + b$ whose length between the two end points (i.e. B_1 on $y = x^2$ and B_2 on $y = x-5$) is the shortest distance (see Problem 1). Moreover, from § 1.9 we should also have two transversality conditions (see Eqs. 24 and 25 noting that in the following we do not use the subscripted version of these equations for simplicity) at the two end points, that is:

$$\begin{aligned}
F+(Y'-y')\,F_{y'} &= 0 && (\text{at point } B_1)\\
\sqrt{1+y'^2}+(Y'-y')\frac{y'}{\sqrt{1+y'^2}} &= 0 && (F=\sqrt{1+y'^2};\ \text{see Problem 1})\\
\sqrt{1+a^2}+(2x-a)\frac{a}{\sqrt{1+a^2}} &= 0 && (y=ax+b \text{ and } Y=x^2)\\
1+a^2+(2x-a)\,a &= 0 && (\times\sqrt{1+a^2})\\
1+a^2+2ax-a^2 &= 0\\
1+2ax_1 &= 0 && (40)
\end{aligned}$$

$$\text{AND}$$

$$\begin{aligned}
F+(Y'-y')\,F_{y'} &= 0 && (\text{at point } B_2)\\
\sqrt{1+y'^2}+(Y'-y')\frac{y'}{\sqrt{1+y'^2}} &= 0 && (F=\sqrt{1+y'^2};\ \text{see Problem 1})\\
\sqrt{1+a^2}+(1-a)\frac{a}{\sqrt{1+a^2}} &= 0 && (y=ax+b \text{ and } Y=x-5)
\end{aligned}$$

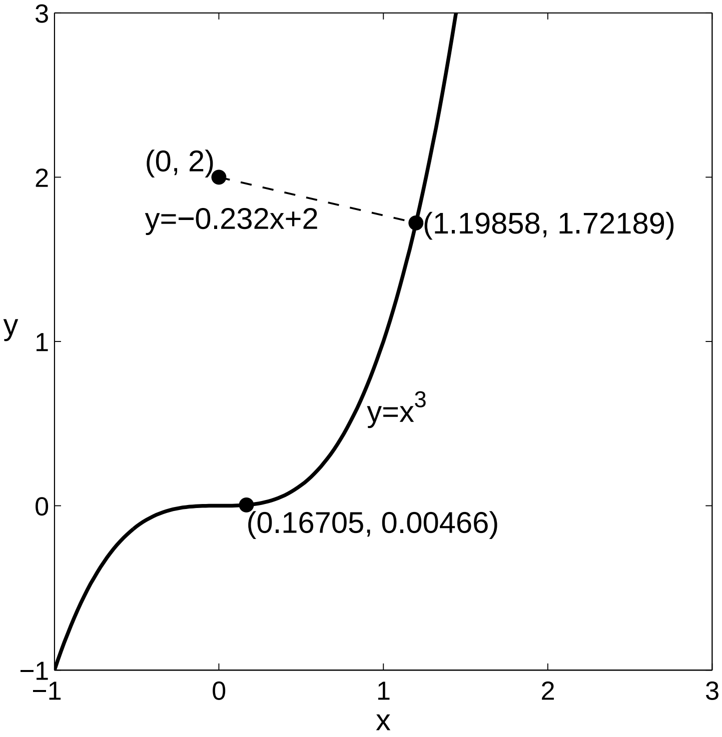

Figure 12: The result of Problem 15 of § 2.1. The point $(0.16705, 0.00466)$ is also plotted for clarity. The plot seems to suggest that d_1 is a local maximum.

$$\begin{aligned} 1 + a^2 + (1-a)\,a &= 0 && (\times\sqrt{1+a^2}) \\ 1 + a^2 + a - a^2 &= 0 \\ a &= -1 \end{aligned} \tag{41}$$

So, from Eq. 41 the equation of the geodesic becomes $y = -x + b$. Also, on substituting from Eq. 41 into Eq. 40 we get $x_1 = 1/2$.
Now, point B_1 is on both the geodesic $y = -x + b$ and the boundary curve $y = x^2$ and hence:

$$-x_1 + b = x_1^2 \qquad \to \qquad -\frac{1}{2} + b = \frac{1}{4}$$

i.e. $b = 3/4$. So, the equation of the geodesic becomes $y = -x + \frac{3}{4}$.
Similarly, B_2 is on both the geodesic $y = -x + \frac{3}{4}$ and the boundary curve $y = x - 5$. Hence:

$$-x_2 + \frac{3}{4} = x_2 - 5 \qquad \to \qquad 2x_2 = 5 + \frac{3}{4}$$

i.e. $x_2 = 23/8$. So in brief, the geodesic is $y = \frac{3}{4} - x$ and the shortest distance $\big[$between point B_1 with coordinates $(\frac{1}{2}, \frac{1}{4})$ and point B_2 with coordinates $(\frac{23}{8}, \frac{-17}{8})\big]$ is (using Pythagoras):

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{\left(\frac{1}{2} - \frac{23}{8}\right)^2 + \left(\frac{1}{4} + \frac{17}{8}\right)^2} = \sqrt{\frac{361}{32}} \simeq 3.35875721$$

For clarity, we plot the result in Figure 13.

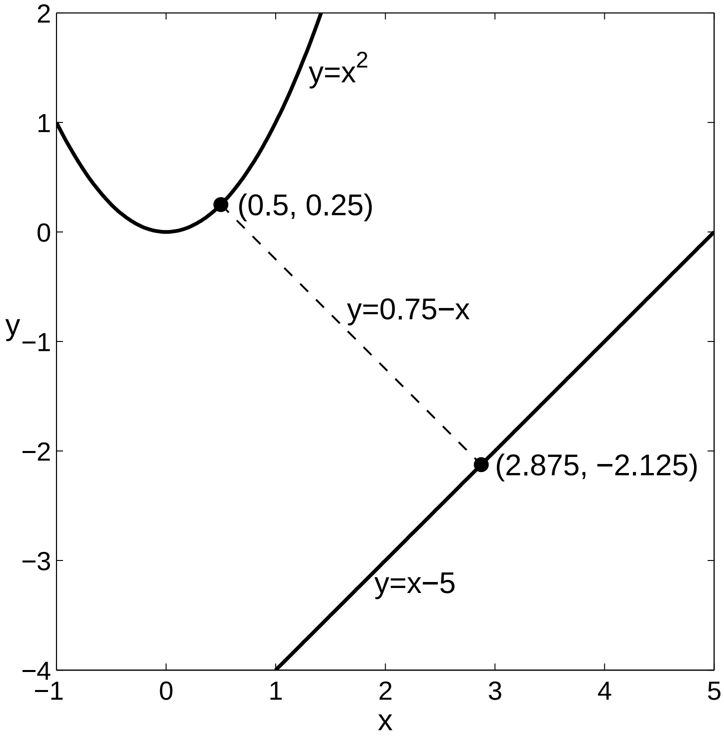

Figure 13: The result of Problem 16 of § 2.1.

17. Find the shortest distance between the parabola $y = x^2 + 2$ and the natural logarithm curve $y = \ln x$.
Answer: This is obviously a geodesic problem (since it is about shortest distance) on a Euclidean plane with two variable boundaries. So, the solution is obviously a straight line $y = ax + b$ whose length between the two end points (i.e. B_1 on $y = x^2 + 2$ and B_2 on $y = \ln x$) is the shortest distance (see Problem 1). Moreover, from § 1.9 we should also have two transversality conditions (see Eqs. 24 and 25 noting that in the following we do not use the subscripted version of these equations) at the two end points, that is:

$$\begin{aligned}
F + (Y' - y')\, F_{y'} &= 0 && \text{(at point B}_1\text{)} \\
\sqrt{1+y'^2} + (Y' - y')\, \frac{y'}{\sqrt{1+y'^2}} &= 0 && (F = \sqrt{1+y'^2};\ \text{see Problem 1}) \\
\sqrt{1+a^2} + (2x - a)\, \frac{a}{\sqrt{1+a^2}} &= 0 && (y = ax + b \text{ and } Y = x^2 + 2) \\
1 + a^2 + (2x - a)\, a &= 0 && (\times\sqrt{1+a^2}) \\
1 + a^2 + 2ax - a^2 &= 0 \\
1 + 2ax_1 &= 0 && (42) \\
&\text{AND} \\
F + (Y' - y')\, F_{y'} &= 0 && \text{(at point B}_2\text{)} \\
\sqrt{1+y'^2} + (Y' - y')\, \frac{y'}{\sqrt{1+y'^2}} &= 0 && (F = \sqrt{1+y'^2};\ \text{see Problem 1}) \\
\sqrt{1+a^2} + \left(x^{-1} - a\right) \frac{a}{\sqrt{1+a^2}} &= 0 && (y = ax + b \text{ and } Y = \ln x) \\
1 + a^2 + \left(x^{-1} - a\right) a &= 0 && (\times\sqrt{1+a^2})
\end{aligned}$$

$$\begin{aligned} 1 + a^2 + ax^{-1} - a^2 &= 0 \\ a &= -x_2 \end{aligned} \tag{43}$$

On substituting from Eq. 43 into Eq. 42 we get:

$$x_2 = \frac{1}{2x_1} \tag{44}$$

Now, point B$_1$ is on both the geodesic $y = ax + b$ and the boundary curve $y = x^2 + 2$ and hence:

$$ax_1 + b = x_1^2 + 2 \qquad \rightarrow \qquad b = x_1^2 - ax_1 + 2 \tag{45}$$

Similarly, B$_2$ is on both the geodesic $y = ax + b$ and the boundary curve $y = \ln x$. Hence:

$$ax_2 + b = \ln x_2 \qquad \rightarrow \qquad b = \ln x_2 - ax_2 \tag{46}$$

On comparing Eq. 45 to Eq. 46 we get:

$$\begin{aligned} x_1^2 - ax_1 + 2 &= \ln x_2 - ax_2 \\ x_1^2 - ax_1 + 2 &= \ln \frac{1}{2x_1} - a\frac{1}{2x_1} && \text{(substituting from Eq. 44)} \\ x_1^2 + x_2x_1 + 2 &= \ln \frac{1}{2x_1} + x_2\frac{1}{2x_1} && \text{(substituting from Eq. 43)} \\ x_1^2 + \frac{1}{2} + 2 &= \ln \frac{1}{2x_1} + \frac{1}{2x_1}\frac{1}{2x_1} && \text{(using Eq. 44)} \\ \ln \frac{1}{2x_1} + \frac{1}{4x_1^2} - x_1^2 - \frac{5}{2} &= 0 \end{aligned}$$

On solving this equation numerically we get $x_1 \simeq 0.335942278$. Hence, $x_2 \simeq 1.488350923$ (from Eq. 44), $a \simeq -1.488350923$ (from Eq. 43), and $b \simeq 2.612857214$ (from Eq. 45 or Eq. 46).
So, the geodesic is $y = -1.488350923x + 2.612857214$ and the shortest distance $\big[$between point B$_1$ with coordinates $(0.335942278, 2.112857214)$ and point B$_2$ with coordinates $(1.488350923, 0.397668744)\big]$ is:

$$\begin{aligned} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} &\simeq \sqrt{(0.335942278 - 1.488350923)^2 + (2.112857214 - 0.397668744)^2} \\ &\simeq 2.066377790 \end{aligned}$$

For clarity, we plot the result in Figure 14.

18. Show that the shortest distance between a point A and a (coplanar) straight line (say Γ_1) is on another straight line (say Γ) which is perpendicular to Γ_1 (see Figure 4 but assume Γ_1 is straight).
Answer: This is obviously a geodesic problem (since it is about shortest distance) on a Euclidean plane with one variable boundary (i.e. Γ_1). So, the solution is obviously a straight line whose length between the point A and the point B (which is on Γ_1) is the shortest distance (see Problem 1). Now, if Γ is given by $y = ax + b$ and Γ_1 is given by $y = \alpha x + \beta$ then we should have a transversality condition (refer to § 1.9 and see Eq. 23) at the point of their intersection (i.e. point B), that is:

$$\begin{aligned} F + (Y' - y')\,F_{y'} &= 0 && \text{(at point B)} \\ \sqrt{1 + y'^2} + (Y' - y')\,\frac{y'}{\sqrt{1 + y'^2}} &= 0 && (F = \sqrt{1 + y'^2};\ \text{see Problem 1}) \\ \sqrt{1 + a^2} + (\alpha - a)\,\frac{a}{\sqrt{1 + a^2}} &= 0 && (y = ax + b \text{ and } Y = \alpha x + \beta) \\ 1 + a^2 + a\alpha - a^2 &= 0 && (\times\sqrt{1 + a^2}) \\ a\alpha &= -1 \end{aligned}$$

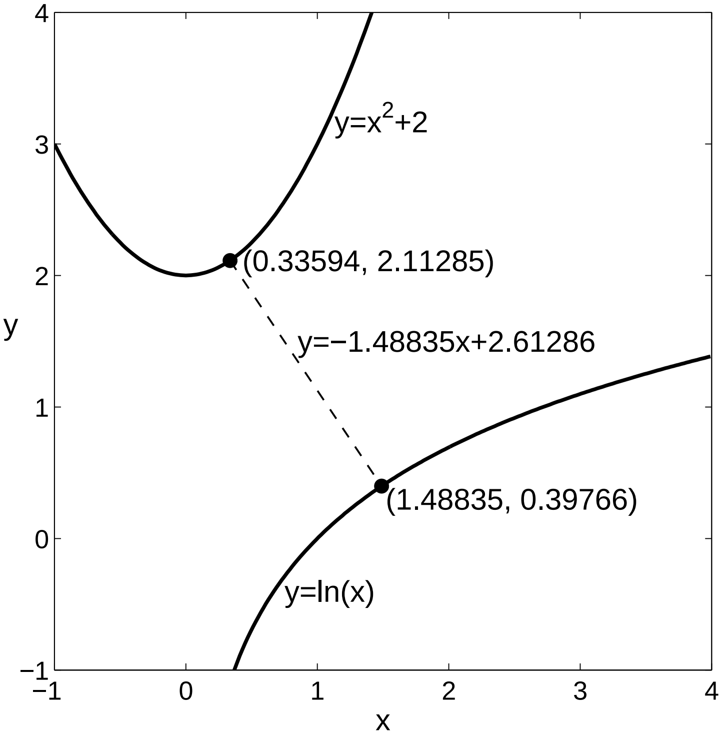

Figure 14: The result of Problem 17 of § 2.1.

As we see, the last equation means that the product of the slopes of Γ and Γ_1 is -1 and hence Γ and Γ_1 are perpendicular.[46] So, we conclude that Γ (whose segment between A and B is of shortest length) is straight and perpendicular to Γ_1, as required.

2.2 Fastest Descent Curves

In this type of variational problems it is required to find the curves along which gravitated objects (i.e. objects in a gravitational field) descend in the shortest time. The most famous of these problems (and indeed the problem that kick-started the calculus of variations as a new branch of mathematics) is the brachistochrone problem.[47] The objective of the problem of brachistochrone in its original and most basic form is to find the curve of fastest descent between two fixed points. More technically, the brachistochrone curve is a planar curve Γ in the vertical plane that connects a point A to a point B (with B being below A but not vertically beneath it) in a uniform gravitational field so that a bead (representing a massive particle) at A takes minimum time to reach B (see Figure 15). The bead is assumed to be released from rest at A and it slides frictionlessly along Γ under the effect of gravity alone. In fact, the brachistochrone problem has a number of variations and modifications some of which (as well as the original problem itself) will be dealt with in the Problems of this section.

We should finally note that apart from the property of being the curve of fastest descent (as implied by its name) the brachistochrone curve has other interesting properties. For example, it can be shown (see Problem 7) that the time taken by a bead to descend to the lowest point (i.e. the point B corresponding

---

[46] It should be known that two straight lines are perpendicular *iff* the product of their slopes equals -1.

[47] As indicated above, the brachistochrone problem marks the beginning of the calculus of variations as an independent branch of modern mathematics. This problem was originally considered by Galileo in 1638 but without reaching a correct solution. The problem was then presented in a technical form and publicized by Johann Bernoulli in the end of the 17^{th} century (around 1696) where solution was obtained by Bernoulli and other contemporary mathematicians. We should finally note that the word “brachistochrone” originates from Greek meaning “shortest time”.

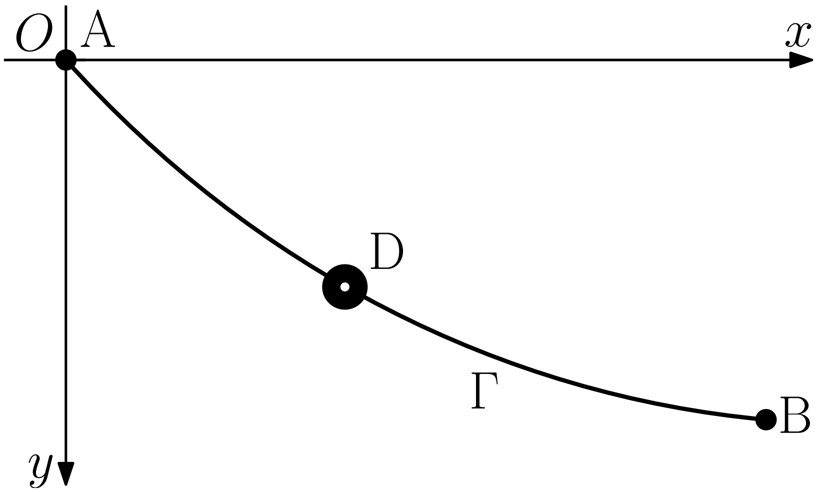

Figure 15: A simple sketch representing a wire Γ that connects two points (A and B) with a bead D sliding frictionlessly along the wire under the effect of gravity alone. The bead is released from rest at point A which is at the origin of an inverted Cartesian coordinate system and hence point A has coordinates $(0, 0)$ while point B has coordinates (x_B, y_B). See § 2.2.

to $\phi = \pi$ as will be clarified in Problem 1) on the curve is independent of the position of the starting point (i.e. the point of release A) on the curve,[48] and hence the brachistochrone curve is also a *tautochrone* (i.e. same time in Greek) or *isochrone* (i.e. equal time in Greek). We should also note that although the fastest descent (or brachistochrone) problem is the most famous of its kind (and hence it is investigated systematically in almost all variational calculus texts) there are many other similar time optimization problems. Some of these problems (such as river crossing) can be found in the textbooks and research papers related to the calculus of variations.

Problems

1. Solve the brachistochrone problem (as described above in the text).
   Answer: Let solve this problem using the setting of Figure 15 where point A is at the origin of an inverted Cartesian coordinate system (i.e. its y axis is pointing downward) in the vertical plane. The bead is initially at rest and hence its kinetic energy is zero (therefore any energy of the bead should be potential). Now, as the bead is released and it starts sliding it loses potential energy by descending into the potential well of the Earth and hence the lost potential energy will be converted to kinetic energy according to the conservation of energy principle (or the work-energy principle). The magnitude of the lost potential energy is given by mgy (where m is the mass of the bead, g is the magnitude of the gravitational field and y is the vertical coordinate of the bead noting that point A is at the origin of coordinates), and hence by the work-energy principle we have (with v being the speed of the bead):

$$\begin{aligned}
\frac{1}{2}mv^2 &= mgy \\
v &= \sqrt{2gy} \\
\frac{ds}{dt} &= \sqrt{2gy} && (v = ds/dt) \\
dt &= \frac{ds}{\sqrt{2gy}} \\
dt &= \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}} && \left[ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1+y'^2}\,dx\right] \qquad (47)
\end{aligned}$$

---

[48] It should be obvious that the starting point is where the bead starts descending (i.e. where it is initially at rest).

$$
\begin{aligned}
\int_0^{t_B} dt &= \int_0^{x_B} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx \\
t_B &= \frac{1}{\sqrt{2g}}\int_0^{x_B} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx \qquad (g \text{ is constant}) \qquad (48)
\end{aligned}
$$

Now, since we are supposed to minimize the time t_B then the integral of the last equation represents our functional integral $I[y]$. So, on comparing the last equation to Eq. 1 we can see that in this case $F \equiv \frac{\sqrt{1+y'^2}}{\sqrt{y}}$ (noting that $1/\sqrt{2g}$ is a constant and hence it is no more than a scaling factor). On applying the Euler-Lagrange equation (noting that F has no explicit dependency on x and hence we can use Eq. 3) we get:

$$
\begin{aligned}
\frac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\frac{\partial}{\partial y'}\left(\frac{\sqrt{1+y'^2}}{\sqrt{y}}\right) &= C \\
\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'}{\sqrt{y}}\frac{2y'}{2\sqrt{1+y'^2}} &= C \\
\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} &= C \\
\frac{1}{\sqrt{y}\sqrt{1+y'^2}} &= C \\
\sqrt{y}\sqrt{1+y'^2} &= D \qquad (D = 1/C) \\
y\left(1+y'^2\right) &= D^2 \\
y'^2 &= \frac{D^2}{y} - 1 \\
y' &= \sqrt{\frac{D^2-y}{y}} \\
\frac{dy}{dx} &= \sqrt{\frac{D^2-y}{y}} \\
dx &= \sqrt{\frac{y}{D^2-y}}\,dy \\
\int_0^{x_B} dx &= \int_0^{y_B} \sqrt{\frac{y}{D^2-y}}\,dy \\
x_B &= \int_0^{y_B} \sqrt{\frac{y}{D^2-y}}\,dy \\
x &= \int_0^{y} \sqrt{\frac{y}{D^2-y}}\,dy \qquad (49)
\end{aligned}
$$

where the last equation is justified by the fact that the curve passes through the origin of coordinates and hence $x_B = x$ and $y_B = y$.[49]

To integrate Eq. 49 we use the substitution $y = D^2\sin^2\theta$ and hence $dy = 2D^2\sin\theta\cos\theta d\theta$. Therefore, Eq. 49 becomes:

$$
x = \int_0^{\theta} \sqrt{\frac{D^2\sin^2\theta}{D^2 - D^2\sin^2\theta}}\,2D^2\sin\theta\cos\theta\,d\theta
$$

---

[49] The use of y as a variable and as a limit of integration should be noticed. This abuse of notation should be tolerated here to avoid unwanted complications.

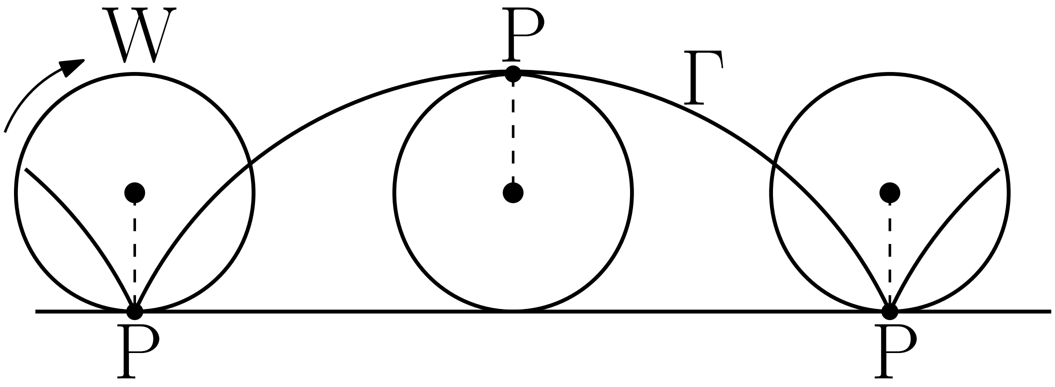

Figure 16: A simple sketch representing a cycloid Γ traced by a point P on the rim of a rolling wheel W. See Problem 1 of § 2.2.

$$\begin{aligned}
x &= \int_0^\theta \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\, 2D^2 \sin\theta\cos\theta\, d\theta \\
x &= \int_0^\theta \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\, 2D^2 \sin\theta\cos\theta\, d\theta \\
x &= \int_0^\theta \frac{\sin\theta}{\cos\theta}\, 2D^2 \sin\theta\cos\theta\, d\theta \\
x &= D^2 \int_0^\theta 2\sin^2\theta\, d\theta \\
x &= D^2 \int_0^\theta (1-\cos 2\theta)\, d\theta \qquad \left[\text{using the identity } \sin^2\theta = (1-\cos 2\theta)/2\right] \\
x &= D^2 \left(\theta - \frac{\sin 2\theta}{2}\right) \\
x &= \frac{D^2}{2}(2\theta - \sin 2\theta)
\end{aligned}$$

Now, if we use $a = D^2/2$ and $\phi = 2\theta$ we get:

$$x = a(\phi - \sin\phi) \qquad \text{and} \qquad y = D^2\sin^2\theta = \left(D^2/2\right) 2\sin^2\theta = a(1-\cos\phi)$$

where we again use the identity $\sin^2\theta = (1-\cos 2\theta)/2$. The last equation (which is a parametric equation of x and y in terms of a constant a and an angular parameter ϕ) represents a cycloid. So, the required brachistochrone curve is a segment of a cycloid.

Note: the cycloid is a curve traced by a point on the rim of a circular wheel (restricted to a plane) as the wheel rolls on a straight line without slipping (see Figure 16; also see Problem 8). We should also note that the optimal time in the brachistochrone problem is a minimum (rather than a maximum) because the descent time along other curves (some of which will be investigated later) is longer.

2. Re-solve the brachistochrone problem (i.e. Problem 1) but this time assume that the bead is not initially at rest.

   Answer: Because the bead in this Problem is not initially at rest, it should have an initial speed v_i. On repeating the analysis of Problem 1, we should again be led to using the work-energy principle (with v being the actual speed of the bead) but with an added term, that is:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_i^2 = mgy$$

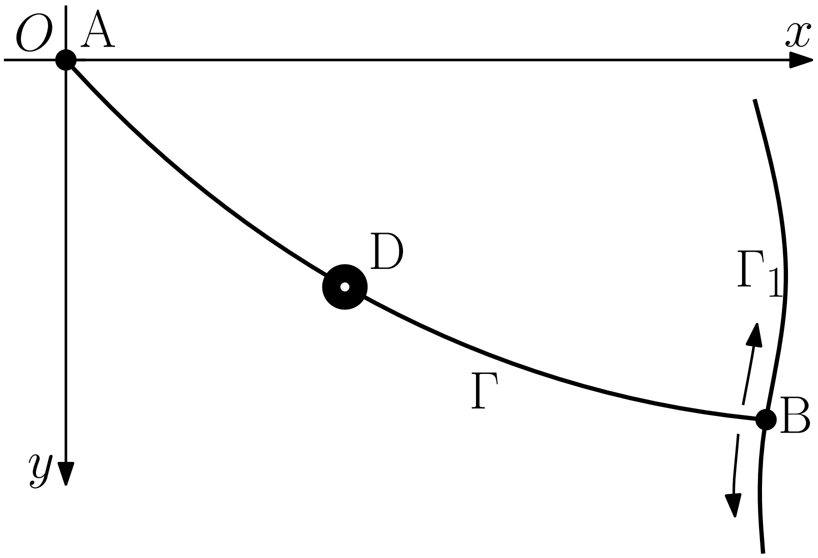

Figure 17: A simple sketch representing a wire Γ connecting a fixed point A to a given curve Γ_1 with a bead D sliding frictionlessly along the wire under the effect of gravity alone. The bead is released from rest at point A which is at the origin of an inverted Cartesian coordinate system. See Problem 3 of § 2.2.

$$\begin{aligned}
v &= \sqrt{2gy + v_i^2} \\
\frac{ds}{dt} &= \sqrt{2gy + v_i^2} \\
dt &= \frac{ds}{\sqrt{2gy + v_i^2}} \\
dt &= \frac{\sqrt{1 + y'^2}\,dx}{\sqrt{2gy + v_i^2}} \\
\int_0^{t_B} dt &= \int_0^{x_B} \frac{\sqrt{1 + y'^2}}{\sqrt{2gy + v_i^2}}\,dx \\
t_B &= \frac{1}{\sqrt{2g}} \int_0^{x_B} \frac{\sqrt{1 + y'^2}}{\sqrt{y + (v_i^2/2g)}}\,dx \\
t_B &= \frac{1}{\sqrt{2g}} \int_0^{x_B} \frac{\sqrt{1 + Y'^2}}{\sqrt{Y}}\,dx
\end{aligned}$$

where Y is defined by the coordinate transformation $Y = y + \frac{v_i^2}{2g}$ (and hence $y' = Y'$). As we see, the last equation is identical in form to Eq. 48 in Problem 1 and hence the solution of this Problem is also a cycloid.

3. Re-solve the problem of fastest descent (i.e. the brachistochrone problem) but assume this time that the bead is sliding from a fixed point A towards a given curve Γ_1 (see Figure 17).
   Answer: In this Problem we are required to find the curve Γ for which the bead (which is released from rest at a fixed point A) reaches the curve Γ_1 in the least possible time. So, this problem is of the type that we investigated in § 1.9 because the curve Γ_1 represents a variable boundary (since the end point which is free to move along this curve is not fixed).
   We use in our solution the same setting as in Problem 1 (see Figure 17). If we follow similar analysis and formulation to the analysis and formulation of Problem 1, then it should be clear that F again is given by $F \equiv \frac{\sqrt{1+y'^2}}{\sqrt{y}}$ and hence the solution here should also be a cycloid connecting the fixed point A

to a yet-unknown point B on the given curve Γ_1. However, we should have an additional transversality condition which is given by Eq. 23, that is:

$$\begin{aligned}
\left[\frac{\sqrt{1+y'^2}}{\sqrt{y}}\right] dX + (dY - y'dX)\frac{\partial}{\partial y'}\left[\frac{\sqrt{1+y'^2}}{\sqrt{y}}\right] &= 0\\
\frac{\sqrt{1+y'^2}}{\sqrt{y}}dX + (dY - y'dX)\frac{y'}{\sqrt{y\left(1+y'^2\right)}} &= 0\\
\frac{1+y'^2}{\sqrt{y\left(1+y'^2\right)}}dX + (dY - y'dX)\frac{y'}{\sqrt{y\left(1+y'^2\right)}} &= 0\\
\frac{dX + y'^2dX + y'dY - y'^2dX}{\sqrt{y\left(1+y'^2\right)}} &= 0\\
\frac{dX + y'dY}{\sqrt{y\left(1+y'^2\right)}} &= 0\\
dX + y'dY &= 0 \qquad (50)\\
y' &= -\frac{dX}{dY}\\
\frac{dy}{dx} &= -\frac{dX}{dY}\\
\frac{dy}{dx}\times\frac{dY}{dX} &= -1
\end{aligned}$$

The last equation means (noting that in the transversality condition y' belongs to Γ at B and dX and dY belong to Γ_1 at B) that the two curves Γ and Γ_1 are perpendicular at B.[50] So in brief, the solution is a cycloid curve Γ connecting the fixed point A to the curve Γ_1 at a point B on Γ_1 such that the curves Γ and Γ_1 are perpendicular to each other at point B.[51]

Note 1: if Γ_1 is a vertical line then the perpendicularity condition requires the slope of Γ at the point of contact to be zero (i.e. the tangent to Γ at point B is horizontal). This can be concluded from Eq. 50 (without going through the subsequent steps) because if Γ_1 is a vertical line then $dX = 0$ (and $dY \neq 0$) and hence Eq. 50 becomes $y'dY = 0$ which leads to $y' = 0$ (since $dY \neq 0$).

Note 2: if Γ_1 is a horizontal line (assuming this case is allowed in the formulation of the brachistochrone problem or it is treated as a separate case) then there are many details and situations to be considered (e.g. whether the point of contact between Γ and Γ_1 is allowed to be vertically beneath the point of release or not). In fact, some of these details should be subject to deliberation and research. However, the simplest situation is when the point of contact is allowed to be beneath the point of release in which case the obvious solution is a vertical straight line (i.e. free fall) where this solution can be seen as a limiting case to the cycloid solution (noting that even this solution meets the perpendicularity condition). Also, see Problem 10.

4. Re-solve the problem of fastest descent (i.e. the brachistochrone problem) but assume this time that the bead is sliding from a given curve Γ_1 towards a fixed point B (see the upper frame of Figure 18).
Answer: For technical reasons (which are fully explained in the literature; see for instance Weinstock in the References), this Problem cannot be solved like Problem 3 by simple application of the above method (with the imposition of the additional transversality condition). However, instead of going through detailed technical analysis and formulation we can obtain the result (which is obtained technically in the literature) by a simple argument, as explained in the following paragraph.
Let the force of gravity become repulsive (instead of being attractive) and hence we are looking for

---

[50] It should be known from elementary calculus that when two curves are perpendicular then the product of their slopes at the intersection point is −1.

[51] This should be understood to set the optimization condition for a generic form of Γ_1. Whether this condition can or cannot be met by a certain form of Γ_1 (defined by certain conditions and restrictions according to the setting of a specific problem) is a different story.

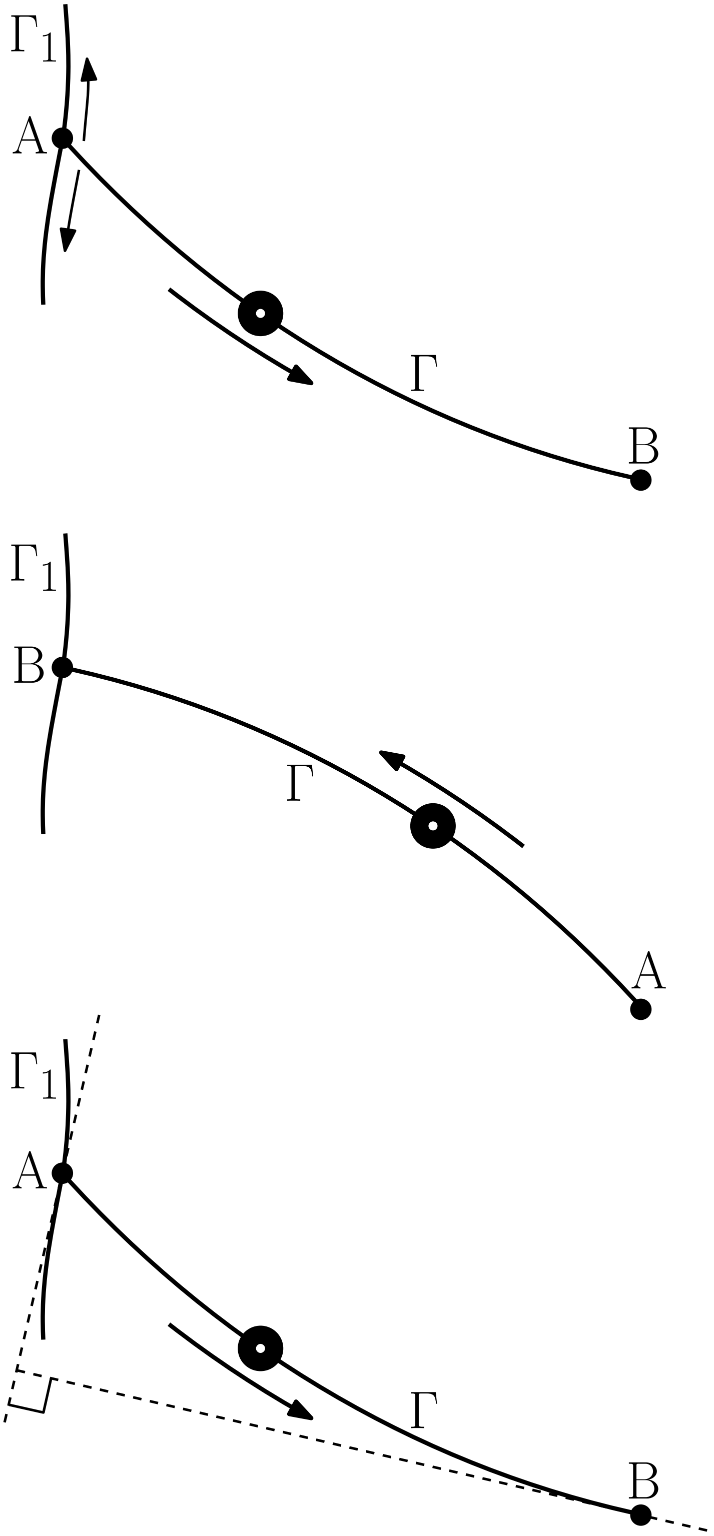

Figure 18: The three frames that demonstrate the setting and reasoning of Problem 4 of § 2.2. The dashed lines in the lower frame are the tangent to Γ_1 at the upper point A and the tangent to Γ at the lower point B.

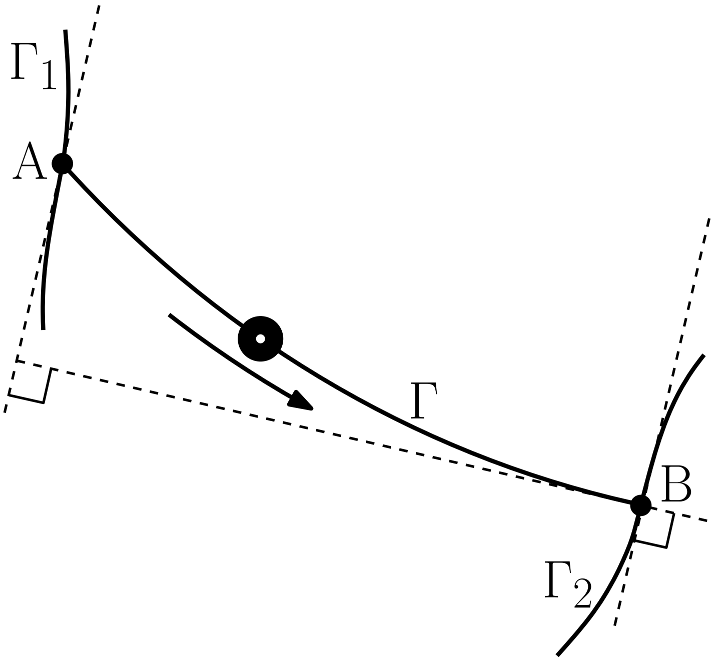

Figure 19: A simple sketch representing the setting and reasoning of Problem 5 of § 2.2.

the curve of fastest *ascent* (which we also label with Γ for simplicity). Accordingly, this Problem will reduce to the setting and formulation of Problem 3 where the bead is sliding from a fixed (lower) point towards (a higher point on) a given curve Γ_1 (see the middle frame of Figure 18) along a curve Γ where this curve (i.e. Γ) is also a cycloid but it is now *concaving downward.* So, according to the result of Problem 3 the curve Γ (in the middle frame of Figure 18) should meet the curve Γ_1 at a point where their tangents are perpendicular. Hence, we obtained the shape of the brachistochrone curve (and in fact the solution) but for the setting of repulsive gravity. Now, let us return to the attractive gravity. In fact, all we need to do now is to rotate the curve Γ through π (i.e. 180°) to match the setting and effect of attractive gravity. Accordingly, the solution of the present Problem is a cycloid (*concaving upward*) whose tangent at the lower point is perpendicular to the tangent of Γ_1 at the upper point (see the lower frame of Figure 18).

5. Re-solve the problem of fastest descent (i.e. the brachistochrone problem) but assume this time that the bead is sliding from a given curve Γ_1 towards a given curve Γ_2 (see Figure 19).
   Answer: If we follow a similar argument to that used in Problem 4 then we can say that the solution of the present Problem can be obtained by combining the results of Problem 3 and Problem 4. In brief, the curve Γ is a (segment of) given cycloid that connects a point A on Γ_1 to a point B on Γ_2. So, from the point of view of connecting point A to curve Γ_2 the perpendicularity condition of Problem 3 applies, while from the point of view of connecting curve Γ_1 to point B the perpendicularity condition of Problem 4 applies. Hence, the solution should be a cycloid that connects points A and B of equal slope (i.e. the slope of Γ_1 at point A is equal to the slope of Γ_2 at point B).
   Note: in fact, there are many details and considerations that should be taken into account in tackling this Problem and establishing its rationale and formulation. So, the above argument is primitive and rough and should be treated as a pedagogical exercise.
6. Referring to the setting of Problem 1, a bead on a cycloid (i.e. cycloid-shaped wire) given by $x = a(\phi - \sin\phi)$ and $y = a(1 - \cos\phi)$ started descending from point $(0, 0)$ at a given time and reached the lowest point on the cycloid at a later time. Find the time required for the bead to descend from point (x_1, y_1) to point (x_2, y_2) where $0 \le x_1 < x_2 \le a\pi$.

Answer: We have:

$$dx = a\left(1-\cos\phi\right)d\phi \qquad dy = a\sin\phi d\phi \qquad y' \equiv \frac{dy}{dx} = \frac{a\sin\phi d\phi}{a\left(1-\cos\phi\right)d\phi} = \frac{\sin\phi}{1-\cos\phi}$$

Now, let t_1 and t_2 correspond to points (x_1, y_1) and (x_2, y_2) respectively (and similarly for ϕ_1 and ϕ_2) and the time required for the bead to descend from point (x_1, y_1) to point (x_2, y_2) be $T = t_2 - t_1$. Starting from Eq. 47, we have:

$$\begin{aligned}
dt &= \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx \\
\int_{t_1}^{t_2} dt &= \int_{x_1}^{x_2}\sqrt{\frac{1+y'^2}{2gy}}\,dx \\
[t]_{t_1}^{t_2} &= \int_{\phi_1}^{\phi_2}\sqrt{\frac{1+\frac{\sin^2\phi}{(1-\cos\phi)^2}}{2ga\left(1-\cos\phi\right)}}\,a\left(1-\cos\phi\right)d\phi \\
t_2 - t_1 &= \int_{\phi_1}^{\phi_2}\sqrt{\frac{\left(1-\cos\phi\right)^2+\sin^2\phi}{2ga\left(1-\cos\phi\right)^3}}\,a\left(1-\cos\phi\right)d\phi \\
T &= \int_{\phi_1}^{\phi_2}\sqrt{\frac{1-2\cos\phi+\cos^2\phi+\sin^2\phi}{2ga\left(1-\cos\phi\right)^3}}\,a\left(1-\cos\phi\right)d\phi \\
T &= \int_{\phi_1}^{\phi_2}\sqrt{\frac{2-2\cos\phi}{2ga\left(1-\cos\phi\right)^3}}\,a\left(1-\cos\phi\right)d\phi \\
T &= \int_{\phi_1}^{\phi_2}\sqrt{\frac{2a^2\left(1-\cos\phi\right)^3}{2ga\left(1-\cos\phi\right)^3}}\,d\phi \\
T &= \int_{\phi_1}^{\phi_2}\sqrt{\frac{a}{g}}\,d\phi \\
T &= \sqrt{\frac{a}{g}}\left(\phi_2-\phi_1\right) \qquad (51)
\end{aligned}$$

Note: as we see, the last equation means that the time of descent between two given points on this curve (assuming the motion started at the origin) is proportional to the angular displacement $(\phi_2 - \phi_1)$. For example, the time of descent from the point corresponding to $\phi_1 = 0$ to the point corresponding to $\phi_2 = \pi/2$ is the same as the time of descent from the point corresponding to $\phi_1 = \pi/2$ to the point corresponding to $\phi_2 = \pi$. Now, if we note that the angular displacement is proportional to the x coordinate of the point of contact of the "wheel" that generates the cycloid (since $x_w = a\phi$ where x_w is the x coordinate of the point of contact of the "wheel") we can imagine the movement of the bead as if it is the result of being attached to the rim of a wheel that moves uniformly (i.e. with constant speed) on the x axis.

7. Show that the brachistochrone curve is also a *tautochrone* (or *isochrone*) curve.
Answer: In this answer we again use the setting of Problem 1. Our objective here is to show that the time taken by a bead to descend to the lowest point $(x_l, y_l) = (a\pi, 2a)$ on the curve is independent of the position of the point of release (from rest) on the curve. Let note first that according to Eq. 51 the time of descent of the bead to the lowest point (i.e. the point corresponding to $\phi_2 = \pi$) on the brachistochrone assuming the bead is released (from rest) at the origin of coordinates (i.e. the point corresponding to $\phi_1 = 0$) is $\pi\sqrt{a/g}$. Now, let assume that instead of releasing the bead (from rest) at the origin of coordinates the bead is released (from rest) at a lower point on the cycloid $\big[$say point

(x_s, y_s) where $0 < x_s < x_l$ and $0 < y_s < y_l\big]$ and hence Eq. 47 becomes:

$$\begin{aligned}
dt &= \frac{\sqrt{1+y'^2}}{\sqrt{2g\,(y-y_s)}}\,dx \\
\int_{t_s}^{t_l} dt &= \int_{x_s}^{x_l}\sqrt{\frac{1+y'^2}{2g\,(y-y_s)}}\,dx \\
[t]_{t_s}^{t_l} &= \int_{\phi_s}^{\phi_l}\sqrt{\frac{1+\frac{\sin^2\phi}{(1-\cos\phi)^2}}{2ga\left([1-\cos\phi]-[1-\cos\phi_s]\right)}}\,a\,(1-\cos\phi)\,d\phi \qquad \text{(see Problem 6)} \\
t_l - t_s &= \int_{\phi_s}^{\pi}\sqrt{\frac{(1-\cos\phi)^2+\sin^2\phi}{2ga\,(\cos\phi_s-\cos\phi)\,(1-\cos\phi)^2}}\,a\,(1-\cos\phi)\,d\phi \\
t_l - t_s &= \int_{\phi_s}^{\pi}\sqrt{\frac{2-2\cos\phi}{2ga\,(\cos\phi_s-\cos\phi)\,(1-\cos\phi)^2}}\,a\,(1-\cos\phi)\,d\phi \\
t_l - t_s &= \int_{\phi_s}^{\pi}\sqrt{\frac{2a^2\,(1-\cos\phi)^3}{2ga\,(\cos\phi_s-\cos\phi)\,(1-\cos\phi)^2}}\,d\phi \\
t_l - t_s &= \int_{\phi_s}^{\pi}\sqrt{\frac{a\,(1-\cos\phi)}{g\,(\cos\phi_s-\cos\phi)}}\,d\phi \\
t_l - t_s &= \sqrt{\frac{a}{g}}\int_{\phi_s}^{\pi}\sqrt{\frac{1-\cos\phi}{\cos\phi_s-\cos\phi}}\,d\phi \\
t_l - t_s &= 2\sqrt{\frac{a}{g}}\int_{\phi_s}^{\pi}\frac{1}{2}\sqrt{\frac{1-\cos\phi}{\cos\phi_s-\cos\phi}}\,d\phi \\
t_l - t_s &= 2\sqrt{\frac{a}{g}}\int_{\phi_s}^{\pi}\frac{\sqrt{\frac{1-\cos\phi}{2}}}{2\sqrt{\frac{\cos\phi_s-\cos\phi}{2}}}\,d\phi \\
t_l - t_s &= 2\sqrt{\frac{a}{g}}\int_{\phi_s}^{\pi}\frac{\sqrt{\frac{1-\cos\phi}{2}}}{2\sqrt{\frac{1+\cos\phi_s}{2}-\frac{1+\cos\phi}{2}}}\,d\phi \\
t_l - t_s &= 2\sqrt{\frac{a}{g}}\int_{\phi_s}^{\pi}\frac{\sin(\phi/2)}{2\sqrt{\cos^2(\phi_s/2)-\cos^2(\phi/2)}}\,d\phi \qquad \text{(trigonometric identities)} \\
t_l - t_s &= 2\sqrt{\frac{a}{g}}\int_{\phi_s}^{\pi}\frac{\sin(\phi/2)}{2\cos(\phi_s/2)\sqrt{\frac{\cos^2(\phi_s/2)-\cos^2(\phi/2)}{\cos^2(\phi_s/2)}}}\,d\phi \\
t_l - t_s &= 2\sqrt{\frac{a}{g}}\int_{\phi_s}^{\pi}\frac{\sin(\phi/2)}{2\cos(\phi_s/2)\sqrt{1-\frac{\cos^2(\phi/2)}{\cos^2(\phi_s/2)}}}\,d\phi \\
t_l - t_s &= 2\sqrt{\frac{a}{g}}\left[-\arcsin\left(\frac{\cos(\phi/2)}{\cos(\phi_s/2)}\right)\right]_{\phi_s}^{\pi} \\
t_l - t_s &= 2\sqrt{\frac{a}{g}}\left[-\arcsin\left(\frac{\cos(\pi/2)}{\cos(\phi_s/2)}\right)+\arcsin\left(\frac{\cos(\phi_s/2)}{\cos(\phi_s/2)}\right)\right] \\
t_l - t_s &= 2\sqrt{\frac{a}{g}}\big[-\arcsin 0+\arcsin 1\big]
\end{aligned}$$

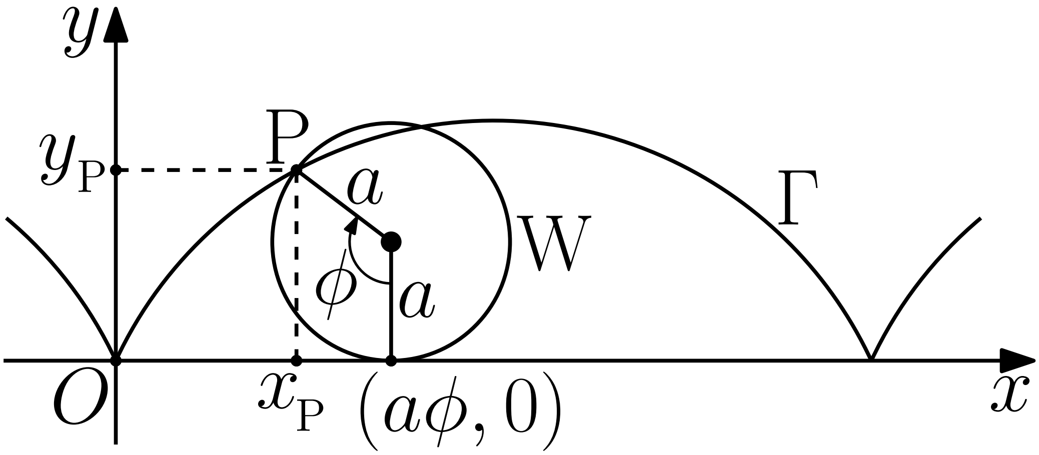

Figure 20: A simple sketch representing a cycloid Γ generated by a point P on the rim of a rolling wheel W of radius a. As can be seen, the x and y coordinates of the point P during the rolling of the wheel are $x_{\mathrm{P}} = a\phi - a\sin\phi$ and $y_{\mathrm{P}} = a - a\cos\phi$. See Problem 8 of § 2.2.

$$\begin{aligned} t_l - t_s &= 2\sqrt{\frac{a}{g}}\left[-0 + \frac{\pi}{2}\right] \\ t_l - t_s &= \pi\sqrt{\frac{a}{g}} \end{aligned}$$

So, according to this equation the time of descent (i.e. $t_l - t_s$) to the lowest point is independent of the position of the release point. In fact, this time is the same as the time of descent if the bead is released from the origin of coordinates (as we noted earlier) which is logical since the release from the origin of coordinates is just a special case of a (general) release point. So, we conclude that the brachistochrone curve is also a *tautochrone* (or *isochrone*) curve, as required.

8. Show that the cycloid is a curve traced by a point on the rim of a circular wheel (restricted to a plane) as the wheel rolls on a straight line without slipping.
   Answer: From the construction of Figure 20 (where the wheel W of radius $r = a$ generates the cycloid Γ) we can easily see that the x and y coordinates of point P (i.e. the point that generates the cycloid) are:

$$\begin{aligned} x_{\mathrm{P}} &= a\phi - a\sin\phi = a(\phi - \sin\phi) \\ y_{\mathrm{P}} &= a - a\cos\phi = a(1 - \cos\phi) \end{aligned}$$

   We note that the signs of the trigonometric functions over the entire range of ϕ (which is of fundamental cycle $0 \le \phi \le 2\pi$) take care of all the possibilities.

9. Using the setting of Problem 1, find the equation of the curve of fastest descent between the points $(0, 0)$ and $(1, 1)$ and plot it.
   Answer: From the result of Problem 1, the curve of fastest descent is a cycloid given by the parametric equations $x = a(\phi - \sin\phi)$ and $y = a(1 - \cos\phi)$. Now, the point $(1, 1)$ should satisfy these equations and hence we have:

$$a(\phi - \sin\phi) = 1 \qquad \text{and} \qquad a(1 - \cos\phi) = 1 \tag{52}$$

   On comparing these equations we get:

$$\begin{aligned} a(\phi - \sin\phi) &= a(1 - \cos\phi) \\ \phi - \sin\phi &= 1 - \cos\phi \end{aligned}$$

On solving this equation (numerically or analytically) we get $\phi \simeq 2.412011144$. On inserting this value of ϕ into one of the equations of Eq. 52 we get $a \simeq 0.572917037$. So, the curve of fastest descent that connects the points $(0,0)$ and $(1,1)$ is given by the parametric equations:

$$x = 0.572917037\,(\phi - \sin\phi) \qquad \text{and} \qquad y = 0.572917037\,(1 - \cos\phi)$$

where $0 \le \phi \le 2\pi$ for a complete cycle. This curve is plotted (for the range $0 \le \phi \le 2\pi$) in Figure 21.

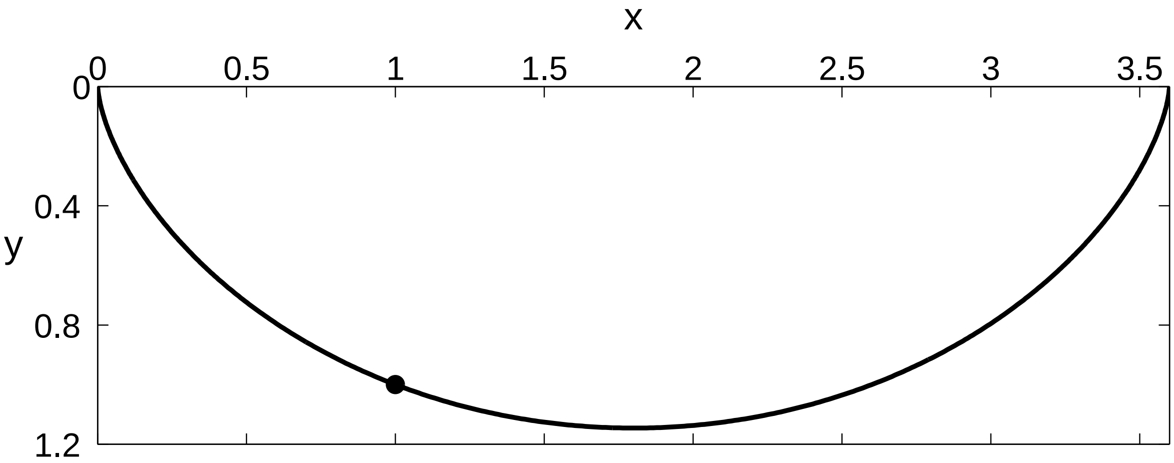

Figure 21: Plot of the curve of fastest descent that connects the points $(0,0)$ and $(1,1)$ for the range $0 \le \phi \le 2\pi$ where the point $(1,1)$ is marked. See Problem 9 of § 2.2.

10. Find the time of descent from point $(0,0)$ to the lowest point on the curve of Problem 9 (assuming SI units). Also, compare the time of descent of this Problem to the time of free fall and the time of descent on the straight line that connects the two points, i.e. point $(0,0)$ and the lowest point.
    Answer: Regarding the time of descent from point $(0,0)$ to the lowest point on the curve of Problem 9, we use Eq. 51 where the point $(0,0)$ corresponds to $\phi_1 = 0$ while the lowest point corresponds to $\phi_2 = \pi$, that is:

$$T = \sqrt{\frac{a}{g}}\,(\phi_2 - \phi_1) = \sqrt{\frac{0.572917037}{9.8}}\,(\pi - 0) \simeq 0.7596\,\text{s}$$

The time of free fall (i.e. vertically from $y_s = 0$ to $y_l = 2a$) is:

$$T = \sqrt{\frac{2}{g}\,(y_l - y_s)} = \sqrt{\frac{2}{9.8}\,(2a - 0)} = \sqrt{\frac{2}{9.8}\,(2 \times 0.572917037)} \simeq 0.4836\,\text{s}$$

Regarding the time of descent on the straight line that connects the two points, we use Eq. 47 (noting that for straight line the slope y' is constant which in our case is $y' = \frac{\Delta y}{\Delta x} = \frac{2a}{a\pi} = \frac{2}{\pi}$ and $y = y'x = \frac{2}{\pi}x$) and hence we have:

$$\begin{aligned}
dt &= \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\,dx \\
dt &= \sqrt{\frac{1 + \frac{4}{\pi^2}}{2g\left(\frac{2}{\pi}x\right)}}\,dx \\
T &= \int_0^{a\pi} \sqrt{\frac{\pi\left(1 + \frac{4}{\pi^2}\right)}{4gx}}\,dx \\
T &= \sqrt{\frac{\pi\left(1 + \frac{4}{\pi^2}\right)}{4g}} \int_0^{a\pi} x^{-1/2}\,dx
\end{aligned}$$

$$\begin{aligned}
T &= \sqrt{\frac{\pi\left(1+\frac{4}{\pi^2}\right)}{4g}}\left[2\sqrt{x}\right]_0^{a\pi} \\
T &= 2\sqrt{a\pi}\sqrt{\frac{\pi\left(1+\frac{4}{\pi^2}\right)}{4g}} \\
T &= \sqrt{\frac{a\pi^2\left(1+\frac{4}{\pi^2}\right)}{g}} \\
T &= \sqrt{\frac{0.572917037\times\pi^2\left(1+\frac{4}{\pi^2}\right)}{9.8}} \\
T &\simeq 0.90046\,\mathrm{s}
\end{aligned}$$

As we see, the shortest is the free fall time, followed by the time of descent on the brachistochrone curve, followed by the time of descent on the straight line.

11. Summarize the main properties of the brachistochrone curve that we obtained in this section.
Answer: We note the following:
• It is the curve of fastest descent between two points (within the stated conditions and considering variant cases). This property is a "consequence" of its characteristic as a brachistochrone (see the text and Problem 1).
• The time of descent on this curve is proportional to the angular displacement (see Problem 6).
• The brachistochrone curve is also a *tautochrone* (i.e. same time) or *isochrone* (i.e. equal time) curve (see Problem 7).

2.3 The Catenary

The objective in the problem of catenary (which originates from a Latin word meaning "chain") is to find the shape of the curve Γ that an inextensible (but flexible) hanging cable or chain (of uniform linear mass density and supported only at its two ends in fixed positions A and B) will take when it is under the influence of (uniform) gravity alone (see Figure 22). As we will show, the catenary has the shape of a hyperbolic cosine curve. In fact, the problem of catenary can be solved by several methods and techniques (including ordinary calculus) as we will see in the Problems.[52]

Problems

1. Solve the problem of catenary as a variational problem (by using the calculus of variations).
Answer: We use a Cartesian coordinate system where the chain is in the xy plane (see Figure 22). In its equilibrium position the chain should take the shape that minimizes its potential energy (where its center of gravity is in the lowest position). Now, the potential energy dU of an infinitesimal segment of length ds and mass dm is $dU = gydm = gy\mu ds$ where g is the magnitude of the gravitational field, y is the height of the segment (say above the ground if we take the ground level as the zero-energy reference level) and μ is the linear mass density of the chain. The potential energy U of the chain is the sum of the potential energies dU of the infinitesimal segments of the chain and hence it is given by the following integral (which is the functional that we intend to extremize in this Problem):

$$U = \int_\Gamma dU = \int_\Gamma gy\mu ds = g\mu\int_\Gamma yds = g\mu\int_{x_A}^{x_B} y\sqrt{1+y'^2}\,dx \equiv I[y] \tag{53}$$

Accordingly, $F\left(x,y,y'\right) = y\sqrt{1+y'^2}$. Since F has no explicit dependency on x we use Eq. 3 (i.e.

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[52] We note that many (if not most) problems in physics and mathematics (including the calculus of variations) can be solved by various methods and techniques and hence the problem of catenary is not an exception in this regard. However, the catenary problem is used here as a case study (due to its simplicity and wide use) for demonstration and practice. Some (few) other Problems in this book are also solved by different methods and techniques for the same reasons.

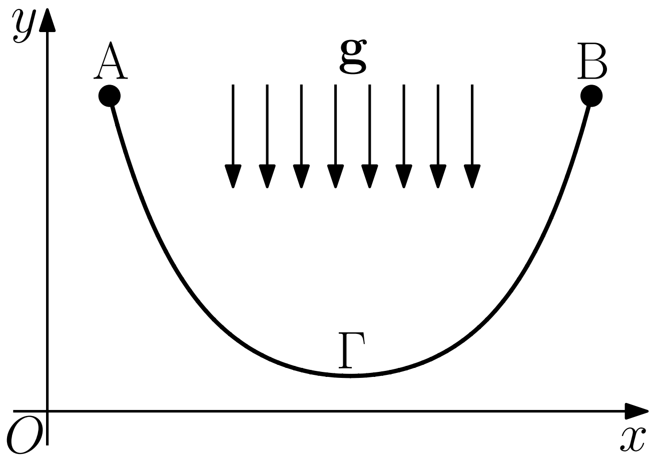

Figure 22: A schematic illustration of the setting of the problem of catenary where a chain Γ of uniform linear density is hanging freely from its two fixed ends (A and B) in a uniform gravitational field **g**. See Problem 1 of § 2.3.

Beltrami identity), that is:

$$\begin{aligned}
\left[y\sqrt{1+y'^2}\right] - y'\frac{\partial}{\partial y'}\left[y\sqrt{1+y'^2}\right] &= C \\
y\sqrt{1+y'^2} - y'\left(y\frac{2y'}{2\sqrt{1+y'^2}}\right) &= C \\
y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} &= C \\
\frac{y+yy'^2}{\sqrt{1+y'^2}} - \frac{yy'^2}{\sqrt{1+y'^2}} &= C \\
\frac{y}{\sqrt{1+y'^2}} &= C \\
\frac{y^2}{1+y'^2} &= C^2 \\
y'^2 &= \frac{y^2}{C^2} - 1 \\
y' &= \sqrt{(y/C)^2 - 1} \\
\frac{dy}{dx} &= \sqrt{(y/C)^2 - 1} \\
dx &= \frac{dy}{\sqrt{(y/C)^2 - 1}} \\
x &= C \operatorname{arccosh}\left(\frac{y}{C}\right) + D \qquad (D \text{ is constant}) \qquad (54)
\end{aligned}$$

On solving the last equation for y we get the profile of the hanging chain, that is:

$$y = C\cosh\left(\frac{x-D}{C}\right) \tag{55}$$

This curve (which has the shape of a hyperbolic cosine) is called catenary. The two constants in this solution (i.e. C and D) can be determined from the two boundary conditions at the two end points of

the curve (see Problem 4).
Note 1: this Problem may be solved rather differently by using the other form of ds (i.e. $ds = \sqrt{1+x'^2}\,dy$), that is:

$$U = \int_\Gamma dU = \int_\Gamma gy\mu ds = g\mu \int_\Gamma yds = g\mu \int_{y_A}^{y_B} y\sqrt{1+x'^2}\,dy \equiv I[x] \tag{56}$$

where the prime means d/dy (see footnote [42]). Accordingly, $F(y,x,x') = y\sqrt{1+x'^2}$. Now, if we use Eq. 2 (with exchange of x and y) then we have:

$$\begin{aligned}
\frac{\partial F}{\partial x} - \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) &= 0 \\
\frac{\partial}{\partial x}\left[y\sqrt{1+x'^2}\right] - \frac{d}{dy}\left(\frac{\partial}{\partial x'}\left[y\sqrt{1+x'^2}\right]\right) &= 0 \\
0 - \frac{d}{dy}\left(\frac{yx'}{\sqrt{1+x'^2}}\right) &= 0 \\
\frac{yx'}{\sqrt{1+x'^2}} &= C \\
y^2x'^2 &= C^2 + C^2x'^2 \\
x'^2 &= \frac{C^2}{y^2 - C^2} \\
\frac{dx}{dy} &= \sqrt{\frac{C^2}{y^2 - C^2}} \\
x &= C\,\text{arccosh}\left(\frac{y}{C}\right) + D
\end{aligned}$$

which is the same as Eq. 54 and hence it leads to Eq. 55.
Note 2: it is obvious that the optimal in this Problem is a minimum (as required by the physics) and not a maximum because other configurations of the chain will increase the potential energy.

2. Re-solve the problem of catenary as a variational problem with constraint (see § 1.8) where the constraint comes from fixing the length of the cable.
Answer: As before (see Problem 1), we have $F = y\sqrt{1+y'^2}$ but we have an added condition which is the constraint $\int G\,dx = \int \sqrt{1+y'^2}\,dx$ and hence:

$$H(x,y,y') \equiv F + \lambda G = y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2} = (y+\lambda)\sqrt{1+y'^2}$$

Now, by a suitable transformation[53] we can bring H to the form $Y\sqrt{1+Y'^2}$ which is the same as the form of F in the catenary problem without constraint (see Eq. 53). Therefore, the solution should also be a hyperbolic cosine (or catenary) as in the case of having no constraint (i.e. Problem 1). In fact, this should be obvious because there is no physical difference between the two cases; the difference is only in the formulation which should not affect the nature of the solution.

3. Re-solve the problem of catenary using this time ordinary calculus (instead of the calculus of variations).
Answer: Referring to Figure 23, a tiny element of the cable should be pulled down by the force of its weight $F_v = g\mu s$ where g is the magnitude of the gravitational field, μ is the linear mass density of the cable and s is the length of the element. This element should also be pulled horizontally (to the left according to the setting of Figure 23) by a horizontal force F_h (where for clarity this force is shown at the lowest point of the cable noting that it is independent of x). Now, this element is in equilibrium and hence there should be a tension force $\mathbf{F}_\tau$ whose horizontal component $|\mathbf{F}_\tau|\cos\phi$ balances F_h while its vertical component $|\mathbf{F}_\tau|\sin\phi$ balances F_v, that is:

$$|\mathbf{F}_\tau|\cos\phi = F_h$$

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[53] If $y = Y - \lambda$ then $y' = Y'$ and hence $H = Y\sqrt{1+Y'^2}$.

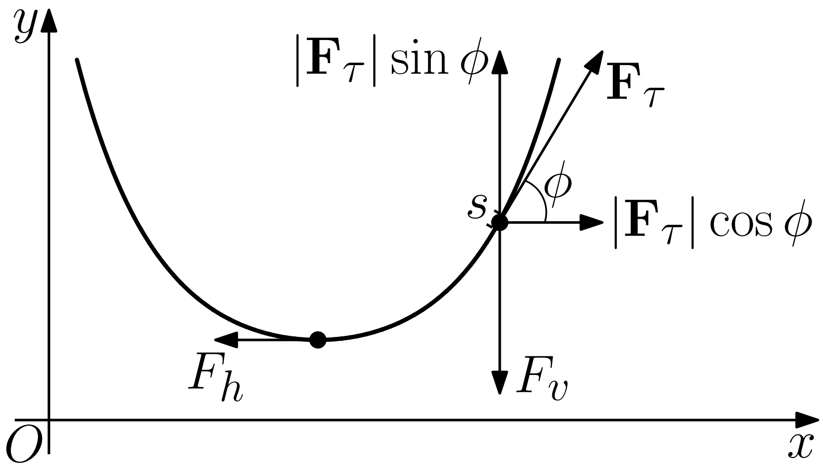

Figure 23: Illustration of the setting of the catenary problem (for the solution by ordinary calculus). See Problem 3 of § 2.3.

$$|\mathbf{F}_\tau|\sin\phi = F_v$$

On dividing the second equation by the first we get:

$$\begin{aligned}
\tan\phi &= \frac{F_v}{F_h} \\
\frac{dy}{dx} &= \frac{g\mu s}{F_h} \\
y' &= \frac{g\mu}{F_h}s \\
\frac{dy'}{dx} &= \frac{g\mu}{F_h}\frac{ds}{dx} \\
\frac{dy'}{dx} &= \frac{g\mu}{F_h}\sqrt{1+y'^2} \\
\frac{dy'}{\sqrt{1+y'^2}} &= \frac{g\mu}{F_h}dx \\
\operatorname{arcsinh}(y') &= \frac{g\mu}{F_h}x+E \qquad\qquad (E \text{ is constant}) \\
y' &= \sinh\left(\frac{g\mu}{F_h}x+E\right) \\
y' &= \sinh\left(\frac{x}{\frac{F_h}{g\mu}}+E\right) \\
y' &= \sinh\left(\frac{x}{\frac{F_h}{g\mu}}+\frac{\frac{F_h}{g\mu}E}{\frac{F_h}{g\mu}}\right) \\
y' &= \sinh\left(\frac{x+\frac{F_h}{g\mu}E}{\frac{F_h}{g\mu}}\right) \\
y &= \frac{F_h}{g\mu}\cosh\left(\frac{x+\frac{F_h}{g\mu}E}{\frac{F_h}{g\mu}}\right) \\
y &= C\cosh\left(\frac{x-D}{C}\right)
\end{aligned}$$

where $C = \frac{F_h}{g\mu}$ and $D = -\frac{F_h}{g\mu}E$. This is the same as Eq. 55 and hence the solution from ordinary calculus is the same as the solution from the calculus of variations (i.e. hyperbolic cosine or catenary).

4. Given that the catenary of Problem 1 passes through the boundary points $(1, 1)$ and $(2, 2)$, find the equation of this catenary and plot it.
   Answer: We solve Eq. 55 (or Eq. 54) for D, that is:

$$D = x - C \operatorname{arccosh}\left(\frac{y}{C}\right)$$

On substituting the coordinates of the two boundary points into this equation we get:

$$D = 1 - C \operatorname{arccosh}\left(\frac{1}{C}\right) \quad (57)$$

$$D = 2 - C \operatorname{arccosh}\left(\frac{2}{C}\right) \quad (58)$$

On subtracting Eq. 57 from Eq. 58 we get:

$$1 - C \operatorname{arccosh}\left(\frac{2}{C}\right) + C \operatorname{arccosh}\left(\frac{1}{C}\right) = 0$$

On solving this equation for C (using a numerical solver) we get $C \simeq 0.949988827$ and hence $D \simeq 0.693083213$ (where we use Eq. 57 or Eq. 58 to find D). Accordingly, the equation of this catenary is:

$$y \simeq 0.949988827 \cosh\left(\frac{x - 0.693083213}{0.949988827}\right) \simeq 0.949988827 \cosh(1.052643959x - 0.729569857)$$

The catenary is plotted in Figure 24.

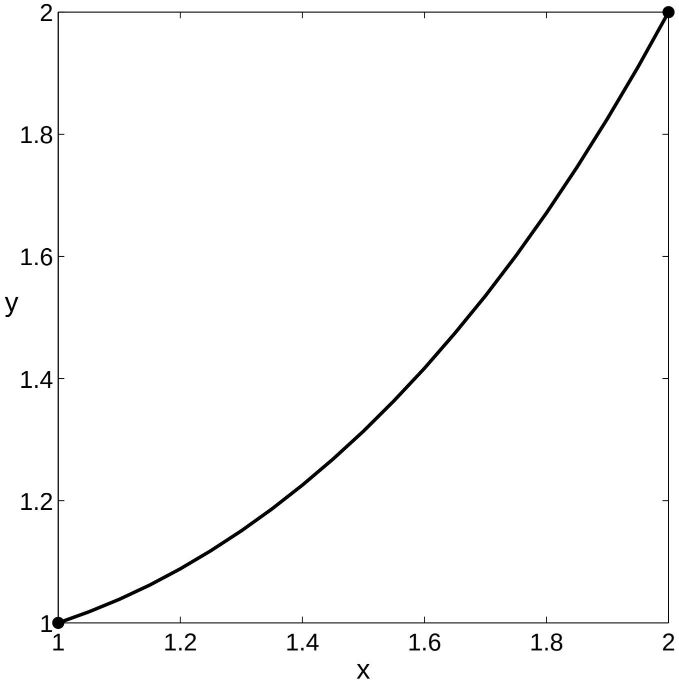

Figure 24: Plot of the (segment of the) catenary that passes through the boundary points $(1, 1)$ and $(2, 2)$. See Problem 4 of § 2.3.

2.4 Isoperimetric Problems

Isoperimetric means "having equal perimeter" and hence these problems primarily deal with situations where we have a perimeter of fixed length that can enclose planar surfaces of different shapes and of different areas. Some authors define isoperimetric problems in a more general way (see for instance Byerly book in the References). In this book we classify under this type of variational problems those problems in which a correlation between length and area is presumed where one imposes a constraint on the other. Anyway, these various classifications, definitions and conventions have very little significance since they have no impact on the mathematics and hence we should not be distracted by these minor issues.

In fact, this type of variational problems includes some of the oldest problems in the history of variational thinking and they actually precede the birth of the modern calculus of variations in the 17^{th} century. The prominent representative of the ancient isoperimetric problems is the Dido problem (which originates from a story in the ancient Greek mythology) where the challenge is to find the planar geometric shape with a perimeter of a given length that encloses maximum area (e.g. what is the shape of a 100 m fence that encloses a flat wheat field of maximum area). In the following we investigate a number of isoperimetric problems (some of which are so simple that they can be solved using ordinary calculus). We should finally note that the classification of isoperimetric problems under the optimal curves (which is the title of the present chapter) may not be ideal but it is of little significance (as indicated earlier).[54]

Problems

1. Find the shape of a planar open curve Γ of a given length l that encloses maximum area between it and a straight line passing through its end points A(x_A, y_A) and B(x_B, y_B).[55]

   Answer: Let the straight line be the x axis and let parameterize the curve by a natural parameter s representing arc length (see Figure 25). Now, the enclosed area (which we want to maximize) is given by the integral:

   $$\sigma = \int d\sigma = \int_{x_A}^{x_B} y\,dx = \int_{s_A}^{s_B} y\sqrt{1-y'^2}\,ds \equiv I$$

   where in step 3 we used $dx = \sqrt{1-y'^2}\,ds$ because $(ds)^2 = (dx)^2 + (dy)^2$ and hence $dx = \sqrt{(ds)^2 - (dy)^2} = \sqrt{1-y'^2}\,ds$ (with the prime representing d/ds). Accordingly, $F(s, y, y') = y\sqrt{1-y'^2}$, and since it is independent of s we use Eq. 3 (noting the correspondence between s and x and the significance of the prime), that is:

   $$\begin{aligned}\left(y\sqrt{1-y'^2}\right) - y'\frac{\partial}{\partial y'}\left(y\sqrt{1-y'^2}\right) &= C\\ y\sqrt{1-y'^2} - y'\left(y\frac{-2y'}{2\sqrt{1-y'^2}}\right) &= C\\ y\sqrt{1-y'^2} + \frac{yy'^2}{\sqrt{1-y'^2}} &= C\\ \frac{y\left(1-y'^2\right) + yy'^2}{\sqrt{1-y'^2}} &= C\\ y &= C\sqrt{1-y'^2}\\ y'^2 &= 1 - \left(\frac{y}{C}\right)^2\end{aligned}$$

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[54] In fact, most of the Problems in the present section are about curves of given length that enclose or border optimal areas although in some Problems we investigated the opposite (i.e. given areas enclosed by curves of optimal length) which may be more appropriate for the classification and inclusion of this section in the "Optimal Curves" chapter.

[55] In this Problem the two end points of the curve are not fixed, i.e. the Problem is based on assuming a loose curve whose end points are free to move on a straight line such that the enclosed area is maximum. In fact, fixing one point and leaving the other free should be sufficient.

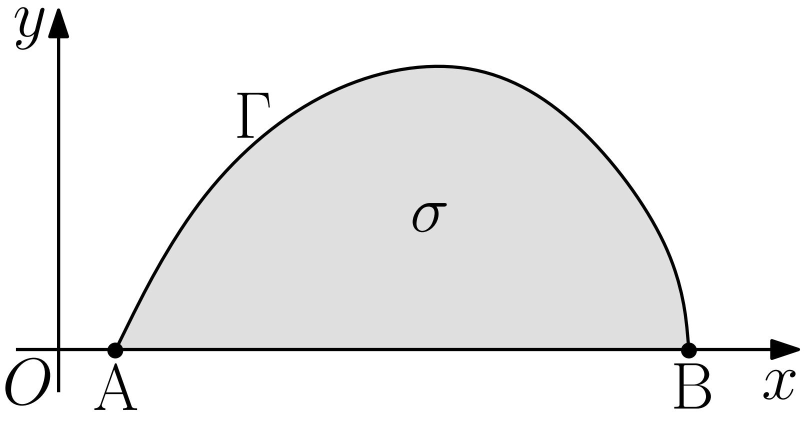

Figure 25: A simple sketch demonstrating a planar curve Γ of a given length l that encloses maximum area σ between it and the x axis. See Problem 1 of § 2.4.

$$\begin{aligned}
\frac{dy}{ds} &= \pm\sqrt{1-\left(\frac{y}{C}\right)^2} \\
\frac{\pm dy}{\sqrt{1-\left(\frac{y}{C}\right)^2}} &= ds \\
\pm C\arcsin\left(\frac{y}{C}\right) &= s+D \\
y &= \pm C\sin\left(\frac{s+D}{C}\right)
\end{aligned}$$

Now, at point A we have $s=0$ and $y=0$ and at point B we have $s=l$ and $y=0$, that is:

$$0=\pm C\sin\left(\frac{D}{C}\right) \qquad \text{and} \qquad 0=\pm C\sin\left(\frac{l+D}{C}\right)$$

that is:

$$\frac{D}{C}=m\pi \qquad \text{and} \qquad \frac{l+D}{C}=\frac{l}{C}+m\pi=n\pi \qquad (m \text{ and } n \text{ are integers})$$

So, $C=\frac{l}{(n-m)\pi}$ and $D=Cm\pi=\frac{lm\pi}{(n-m)\pi}=\frac{lm}{(n-m)}$. Now, to simplify the solution we make $n-m=1$ (because m and n are arbitrary) and make $D=0$ (because it is just a shift in s and hence it is also arbitrary).[56] We also note that the curve is above the x axis (and hence y is positive). Accordingly, the solution becomes:

$$y=\frac{l}{\pi}\sin\left(\frac{\pi s}{l}\right) \qquad (0\le s\le l)$$

Now, we have:

$$\begin{aligned}
\frac{dy}{ds} &= \cos\left(\frac{\pi s}{l}\right) \\
\left(\frac{dy}{ds}\right)^2 &= \cos^2\left(\frac{\pi s}{l}\right) \\
1-\left(\frac{dx}{ds}\right)^2 &= \cos^2\left(\frac{\pi s}{l}\right) \qquad \left[(ds)^2=(dx)^2+(dy)^2\right]
\end{aligned}$$

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[56] In fact, this may be seen as choosing a specific solution more than a simplification although this should not affect the generality of the final result.

$$\begin{aligned}
\left(\frac{dx}{ds}\right)^2 &= 1-\cos^2\left(\frac{\pi s}{l}\right) \\
\left(\frac{dx}{ds}\right)^2 &= \sin^2\left(\frac{\pi s}{l}\right) \\
\frac{dx}{ds} &= \pm\sin\left(\frac{\pi s}{l}\right) \\
x &= \mp\frac{l}{\pi}\cos\left(\frac{\pi s}{l}\right)+E \qquad\qquad (E \text{ is constant}) \\
x-E &= \mp\frac{l}{\pi}\cos\left(\frac{\pi s}{l}\right)
\end{aligned}$$

Therefore:

$$\begin{aligned}
(x-E)^2+y^2 &= \left(\frac{l}{\pi}\right)^2\cos^2\left(\frac{\pi s}{l}\right)+\left(\frac{l}{\pi}\right)^2\sin^2\left(\frac{\pi s}{l}\right) \\
(x-E)^2+y^2 &= \left(\frac{l}{\pi}\right)^2\left[\cos^2\left(\frac{\pi s}{l}\right)+\sin^2\left(\frac{\pi s}{l}\right)\right] \\
(x-E)^2+y^2 &= \left(\frac{l}{\pi}\right)^2 \qquad (59)
\end{aligned}$$

which is an equation of a circle with center $(E, 0)$ and radius l/π. Accordingly, the curve has the shape of a circular arc. In fact, the curve is a semi-circle of length l and center $(E, 0)$.
Note: the optimal area in this Problem is a maximum (not a minimum) because the enclosed area can approach zero when the curve Γ approaches the straight line (and hence Γ becomes effectively straight). The enclosed area can also approach zero in another extreme case when the two end points of the curve approach each other and the two sides of the curve also approach each other. This result will also be confirmed in Problem 2 (assuming the curve to be a circular arc which is the result that we obtained in the present Problem). We should also note that if the curve is shaped as a complete circle (i.e. with A and B being the same point) then its area (which is $\sigma=\frac{l^2}{4\pi}$) is half the area of the semi-circle (which is $\sigma=\frac{l^2}{2\pi}$).

2. Assuming that the curve in Problem 1 is a circular arc, show formally that it is a semi-circle.[57]
Answer: The area of a segment of a circle is given by:

$$\sigma=\frac{1}{2}(\theta-\sin\theta)r^2=\frac{1}{2}(\theta-\sin\theta)\frac{l^2}{\theta^2}=\frac{l^2}{2}\left(\frac{1}{\theta}-\frac{\sin\theta}{\theta^2}\right)$$

where in the second step we used $r=l/\theta$ (since $l=r\theta$ with r being the radius of the circle, θ being the central angle subtending the segment, and l being the length of the circular arc subtending θ). Now, to find the optimal value of σ we take its derivative with respect to θ (since it is the variable noting that l is fixed) and set it to zero to obtain the optimization condition, that is:

$$\begin{aligned}
\frac{2}{l^2}\frac{d\sigma}{d\theta} &= 0 \\
-\frac{1}{\theta^2}-\frac{\cos\theta}{\theta^2}+2\frac{\sin\theta}{\theta^3} &= 0 \\
\theta+\theta\cos\theta-2\sin\theta &= 0
\end{aligned}$$

The solution of this equation in the interval $(0, 2\pi]$ is $\theta=\pi$, i.e. the segment of optimal area subtends an angle $\theta=\pi$ which means that the circular arc is a semi-circle and the segment is a semi-disc. This result can also be confirmed graphically by plotting σ as a function of θ (see Figure 26 where σ is given in units of l^2).

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[57] Noting the setting of Problem 1, Eq. 59 should be sufficient for this demonstration. So, the present Problem is about another way for the required demonstration.

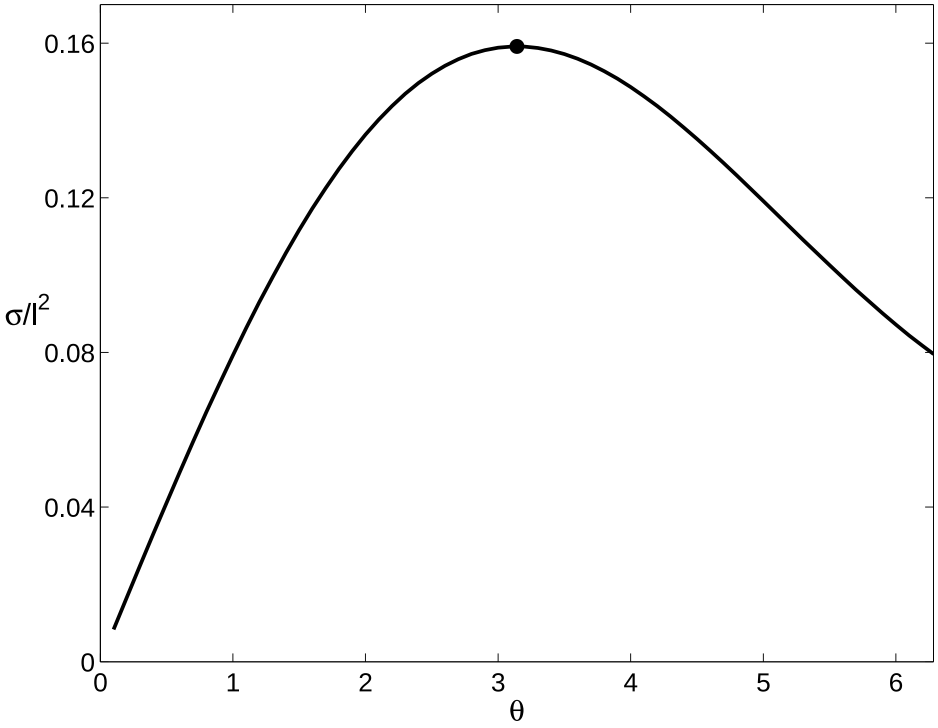

Figure 26: Plot of σ (in units of l^2) as a function of θ where the peak of the curve at $(\pi, \frac{1}{2\pi})$ is marked. See Problem 2 of § 2.4.

We should finally note that we can show formally that the optimal area in this Problem is a maximum (not a minimum) by employing the second derivative test, that is:

$$\begin{aligned}
\left.\frac{d^2\sigma}{d\theta^2}\right|_{\theta=\pi} &= \left.\frac{l^2}{2}\left(\frac{2}{\theta^3} + \frac{2\cos\theta}{\theta^3} + \frac{\sin\theta}{\theta^2} - \frac{6\sin\theta}{\theta^4} + \frac{2\cos\theta}{\theta^3}\right)\right|_{\theta=\pi} \\
&= \frac{l^2}{2}\left(\frac{2}{\pi^3} + \frac{2\cos\pi}{\pi^3} + \frac{\sin\pi}{\pi^2} - \frac{6\sin\pi}{\pi^4} + \frac{2\cos\pi}{\pi^3}\right) \\
&= \frac{l^2}{2}\left(\frac{2}{\pi^3} - \frac{2}{\pi^3} + 0 - 0 - \frac{2}{\pi^3}\right) \\
&= -\frac{l^2}{\pi^3} < 0
\end{aligned}$$

Hence, the optimal area is a maximum.

3. Given that the length of the curve in Problem 1 is $l = 5$, find the maximum enclosed area.
Answer: As seen in Problems 1 and 2, the curve is a semi-circle of radius $r = \frac{l}{\pi}$ and hence the maximum enclosed area is:

$$\sigma = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\frac{l^2}{\pi^2} = \frac{l^2}{2\pi} = \frac{5^2}{2\pi} = \frac{25}{2\pi} \simeq 3.97887$$

4. Find the shape of a planar curve Γ of a given length l that has maximum area beneath it (and above the x axis) with two fixed end points A and B (see Figure 27).
Answer: We are supposed to maximize the area beneath the curve subject to the constraint that the length of the curve is equal to l and hence we use the Lagrange multipliers technique (see § 1.8). Now, the area is given by the integral $\int_{x_1}^{x_2} y\,dx$ while the length of the curve is given by the integral

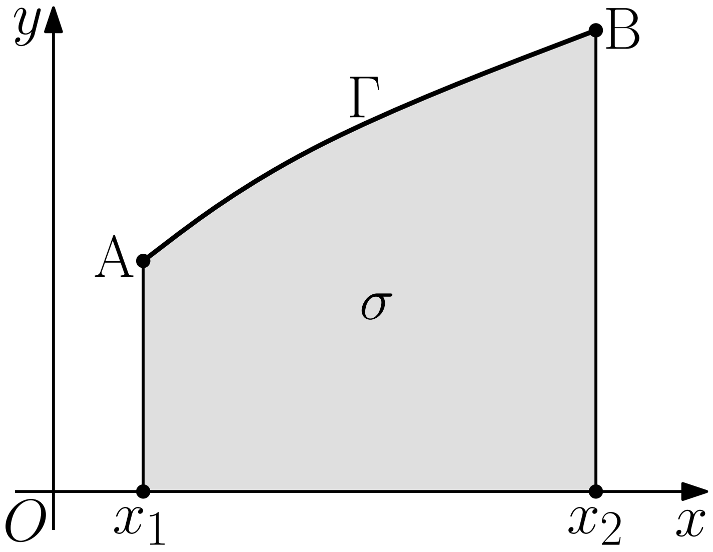

Figure 27: A simple sketch demonstrating a planar curve Γ of a given length l (connecting two fixed points A and B) that has maximum area σ beneath it. See Problem 4 of § 2.4.

$\int_\Gamma \sqrt{(dx)^2 + (dy)^2} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx$. So, $F = y$ and $G = \sqrt{1 + y'^2}$ and hence $H \equiv F + \lambda G = y + \lambda\sqrt{1 + y'^2}$. Using Eq. 22, we have:[58]

$$\begin{aligned}
\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) &= 0\\
\frac{\partial}{\partial y}\left(y + \lambda\sqrt{1+y'^2}\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[y + \lambda\sqrt{1+y'^2}\right]\right) &= 0\\
1 - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) &= 0\\
\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) &= 1\\
\frac{\lambda y'}{\sqrt{1+y'^2}} &= x + C_1\\
\lambda^2 y'^2 &= (x + C_1)^2\left(1 + y'^2\right)\\
y'^2\left[\lambda^2 - (x + C_1)^2\right] &= (x + C_1)^2\\
y' &= \frac{x + C_1}{\sqrt{\lambda^2 - (x + C_1)^2}} \qquad (60)\\
y &= -\sqrt{\lambda^2 - (x + C_1)^2} + C_2\\
(x + C_1)^2 + (y - C_2)^2 &= \lambda^2 \qquad (61)
\end{aligned}$$

As we see, the last line is the equation of a circular arc with center $(-C_1, C_2)$ and radius $|\lambda|$.
Note 1: the three parameters C_1, C_2, λ in the above solution can be determined from the two boundary conditions (i.e. the coordinates of the end points A and B) plus the constraint on the length l (see

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[58] In fact, we can also us the Beltrami identity (Eq. 3) with H replacing F.

Problem 5).

Note 2: assuming the curve to concave downward the optimal area is a maximum (not a minimum) because the area can become a minimum when the curve concaves upward. In fact, the technicalities of the problem (supported by the associated conditions) should take care of these issues.

Note 3: we are assuming certain restrictions on the length of the curve (i.e. limits on its minimum and maximum) for this Problem and its solution to apply. The details can be easily worked out in each particular problem.

5. Given that the curve in Problem 4 passes through the points $(2, 4)$ and $(5, 1)$ and the length of the arc is $\frac{3\pi}{2}$, determine the unknown constants in the solution and hence obtain the specific solution. Also, plot the curve and find the area beneath the arc.

Answer: On inserting the coordinates of the points $(2, 4)$ and $(5, 1)$ into Eq. 61 we get:

$$\begin{aligned}(2+C_1)^2+(4-C_2)^2 &= \lambda^2 &\quad (62)\\ (5+C_1)^2+(1-C_2)^2 &= \lambda^2 &\quad (63)\end{aligned}$$

On subtracting Eq. 63 from Eq. 62 we get:

$$\begin{aligned}(2+C_1)^2+(4-C_2)^2-(5+C_1)^2-(1-C_2)^2 &= 0\\ 4+4C_1+C_1^2+16-8C_2+C_2^2-25-10C_1-C_1^2-1+2C_2-C_2^2 &= 0\\ -6-6C_1-6C_2 &= 0\\ C_2 &= -1-C_1 &\quad (64)\end{aligned}$$

Also, by adding Eq. 63 to Eq. 62 we get:

$$\begin{aligned}(2+C_1)^2+(4-C_2)^2+(5+C_1)^2+(1-C_2)^2 &= 2\lambda^2\\ 4+4C_1+C_1^2+16-8C_2+C_2^2+25+10C_1+C_1^2+1-2C_2+C_2^2 &= 2\lambda^2\\ 46+14C_1+2C_1^2-10C_2+2C_2^2 &= 2\lambda^2\\ 23+7C_1+C_1^2-5C_2+C_2^2 &= \lambda^2\\ 23+7C_1+C_1^2-5(-1-C_1)+(-1-C_1)^2 &= \lambda^2 &\quad \text{(from Eq. 64)}\\ 23+7C_1+C_1^2+5+5C_1+1+2C_1+C_1^2 &= \lambda^2\\ 29+14C_1+2C_1^2 &= \lambda^2 &\quad (65)\end{aligned}$$

Now, the length of the curve is given by $l = \int_2^5 \sqrt{1+y'^2}\,dx$ where y' is given by Eq. 60, that is:

$$\begin{aligned} l &= \int_2^5 \sqrt{1+\frac{(x+C_1)^2}{\lambda^2-(x+C_1)^2}}\,dx\\ \frac{3\pi}{2} &= \int_2^5 \sqrt{\frac{\lambda^2-(x+C_1)^2+(x+C_1)^2}{\lambda^2-(x+C_1)^2}}\,dx\\ \frac{3\pi}{2} &= \int_2^5 \sqrt{\frac{\lambda^2}{\lambda^2-(x+C_1)^2}}\,dx\\ \frac{3\pi}{2} &= \int_2^5 \sqrt{\frac{1}{1-(\frac{x+C_1}{\lambda})^2}}\,dx\\ \frac{3\pi}{2} &= \left[\lambda \arcsin\left(\frac{x+C_1}{\lambda}\right)\right]_2^5\\ \frac{3\pi}{2} &= \lambda\left[\arcsin\left(\frac{5+C_1}{\lambda}\right)-\arcsin\left(\frac{2+C_1}{\lambda}\right)\right]\end{aligned}$$

$$\frac{3\pi}{2} = \sqrt{29+14C_1+2C_1^2}\left[\arcsin\left(\frac{5+C_1}{\sqrt{29+14C_1+2C_1^2}}\right) - \arcsin\left(\frac{2+C_1}{\sqrt{29+14C_1+2C_1^2}}\right)\right]$$

where in the last equation we substituted for λ from Eq. 65. On using a numerical solver we find $C_1 = -2$ and hence $C_2 = 1$ (from Eq. 64) and $\lambda = 3$ (from Eq. 65). Therefore, the specific solution is $(x-2)^2 + (y-1)^2 = 9$. The circle is plotted in Figure 28. Regarding the area beneath the arc, it is obvious that the central angle that subtends the arc $\big[$i.e. the angle between the points $(2,4)$, $(2,1)$ and $(5,1)\big]$ is a right angle and hence the area of this sector is one quarter of the area of the circle, i.e. it is $\frac{9\pi}{4}$. We should also add the area of the rectangle $\big[$whose vertices are the points $(2,0)$, $(5,0)$, $(5,1)$ and $(2,1)\big]$ which is equal to $1 \times 3 = 3$. Therefore, the area beneath the arc is $\sigma = \frac{9\pi}{4} + 3 \simeq 10.06858$.

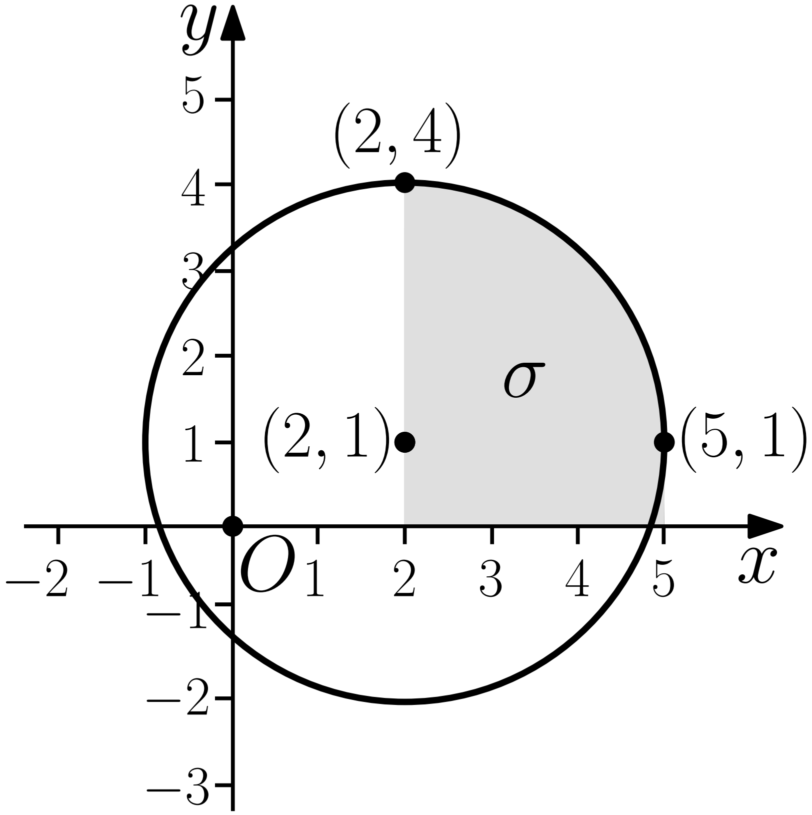

Figure 28: Plot of the circle $(x-2)^2 + (y-1)^2 = 9$ where the area σ under the circular arc is shown in gray. See Problem 5 of § 2.4.

6. Find the shape of the closed plane curve of a given length that encloses maximum area.
Answer: We use a 2D Cartesian system (in the plane of the curve) whose origin O is inside the curve (see Figure 29). Now, an infinitesimal sector $O\overset{\frown}{\mathrm{AB}}$ can be represented by a triangle $O\mathrm{AB}$ and hence its area is given by:

$$d\sigma = \frac{1}{2}\left|\mathbf{r} \times d\mathbf{r}\right| = \frac{1}{2}\left|(x,y,0) \times (dx,dy,0)\right| = \frac{1}{2}\left(x\,dy - y\,dx\right)$$

Therefore, the area inside the curve is:

$$\sigma = \oint d\sigma = \oint \frac{1}{2}\left(x\,dy - y\,dx\right) = \oint \frac{1}{2}\left(xy' - y\right)\,dx$$

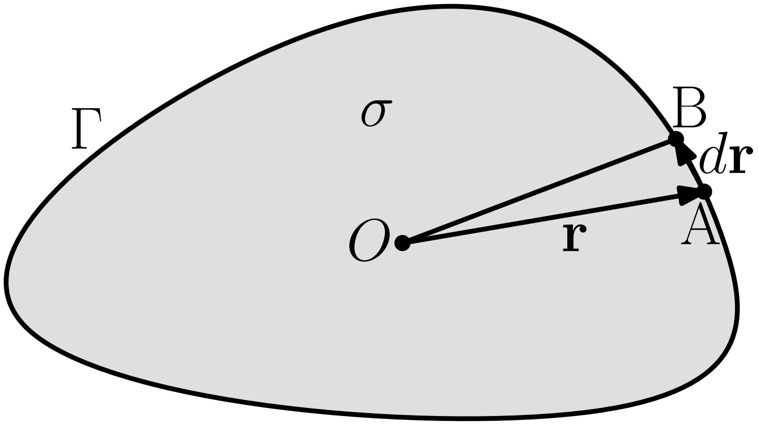

Figure 29: A simple sketch representing a closed planar curve Γ enclosing an area σ. See Problem 6 of § 2.4.

Similarly, the length of the curve is given by the integral $\oint \sqrt{(dx)^2 + (dy)^2} = \oint \sqrt{1 + y'^2}\, dx$.
Now, we are required to maximize the area subject to the constraint that the length is constant. So, we use the Lagrange multipliers technique (see § 1.8) where $H \equiv F + \lambda G = \frac{1}{2}(xy' - y) + \lambda\sqrt{1 + y'^2}$. Using the Euler-Lagrange equation (Eq. 22), we have:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[\frac{1}{2}(xy' - y) + \lambda\sqrt{1 + y'^2}\right] - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[\frac{1}{2}(xy' - y) + \lambda\sqrt{1 + y'^2}\right]\right) &= 0 \\
-\frac{1}{2} - \frac{d}{dx}\left(\frac{x}{2} + \frac{\lambda y'}{\sqrt{1 + y'^2}}\right) &= 0 \\
\frac{d}{dx}\left(\frac{x}{2} + \frac{\lambda y'}{\sqrt{1 + y'^2}}\right) &= -\frac{1}{2} \\
\frac{x}{2} + \frac{\lambda y'}{\sqrt{1 + y'^2}} &= -\frac{x}{2} + C \\
\frac{\lambda y'}{\sqrt{1 + y'^2}} &= C - x \\
\lambda^2 y'^2 &= (C - x)^2 \left(1 + y'^2\right) \\
\lambda^2 y'^2 - (C - x)^2 y'^2 &= (C - x)^2 \\
y'^2 &= \frac{(C - x)^2}{\lambda^2 - (C - x)^2} \\
y' &= \pm\frac{(C - x)}{\sqrt{\lambda^2 - (C - x)^2}} \\
y &= \pm\sqrt{\lambda^2 - (C - x)^2} + D \\
(C - x)^2 + (y - D)^2 &= \lambda^2 \\
(x - C)^2 + (y - D)^2 &= \lambda^2
\end{aligned}$$

This is an equation of a circle with center (C, D) and radius $|\lambda|$. So, the shape of the curve that maximizes the enclosed area is a circle (noting that the solution cannot be a minimum because the area enclosed by a squeezed shape can approach zero). In fact, this solution is also intuitive and hence it was reached even by the ancient scholars using very simple arguments and reasoning.

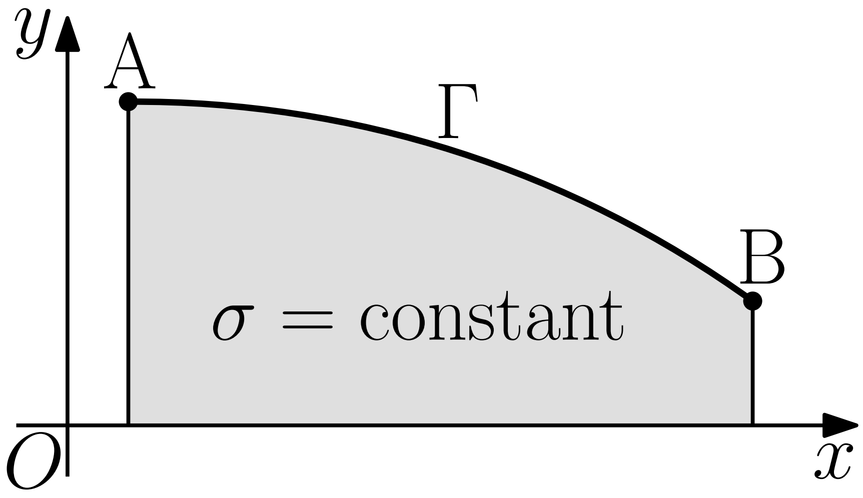

Figure 30: A simple sketch demonstrating the setting of Problem 8 of § 2.4 where a planar curve Γ of shortest length (connecting two fixed points A and B) encloses fixed area σ between it and the x axis.

7. Find the equation of the closed plane curve of length $l \simeq 4.18879$ that encloses maximum area σ and passes through the points $(-0.5, -0.83333)$ and $(0, -1.05904)$. Also find the enclosed area.
   Answer: From the answer of Problem 6 we know that this curve is a circle of radius λ and circumference l. Accordingly, $l = 4.18879 = 2\pi\lambda$ and hence $\lambda = 0.66667 = 2/3$. Therefore, the equation of the curve becomes $(x - C)^2 + (y - D)^2 = \frac{4}{9}$. On substituting the coordinates of the points $(-0.5, -0.83333)$ and $(0, -1.05904)$ into this equation we get:

$$\begin{aligned}(-0.5 - C)^2 + (-0.83333 - D)^2 &= \frac{4}{9} \\ C^2 + (-1.05904 - D)^2 &= \frac{4}{9}\end{aligned}$$

   On solving these equations simultaneously (e.g. by substitution from the second into the first) we get $C = -0.5$ and $D = -1.5$.[59] So, the equation of the curve is $(x + 0.5)^2 + (y + 1.5)^2 = \frac{4}{9}$. The enclosed area is $\sigma = \pi\lambda^2 = \frac{4\pi}{9} \simeq 1.396263$.
8. Find the shape of the planar curve of shortest length that connects two given points $\left[\text{say point } (x_1, y_1)\right.$ and point $\left.(x_2, y_2)\right]$ such that the area beneath it (and above the x axis) is a given constant (see Figure 30).
   Answer: We are required to minimize the arc length s of the curve Γ which is represented by the function $y = y(x)$ subject to the area constraint $\sigma = a$ (where a is a given constant). Now, the length is given by $s = \int_\Gamma ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \equiv I_1$ while the area is given by $\sigma = \int_{x_1}^{x_2} y\, dx \equiv I_2$. So, on using the Lagrange multipliers technique (see § 1.8) with $F \equiv \sqrt{1 + y'^2}$ and $G \equiv y$ we have $H = F + \lambda G = \sqrt{1 + y'^2} + \lambda y$. Hence, the Euler-Lagrange equation (noting that H is independent of x and hence we can use Eq. 3 with H replacing F) is:

$$\begin{aligned}H - y'\frac{\partial H}{\partial y'} &= C \\ \left[\sqrt{1 + y'^2} + \lambda y\right] - y'\frac{\partial}{\partial y'}\left[\sqrt{1 + y'^2} + \lambda y\right] &= C \\ \sqrt{1 + y'^2} + \lambda y - y'\frac{y'}{\sqrt{1 + y'^2}} &= C\end{aligned}$$

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[59] In fact, there is another solution (i.e. $C = 0$ and $D \simeq -0.392373$) which we do not consider here.

$$\begin{aligned}
1+y'^2+\lambda y\sqrt{1+y'^2}-y'^2 &= C\sqrt{1+y'^2}\\
1+\lambda y\sqrt{1+y'^2} &= C\sqrt{1+y'^2}\\
\sqrt{1+y'^2} &= \frac{-1}{\lambda y - C}\\
1+y'^2 &= \frac{1}{(\lambda y - C)^2}\\
y' &= \pm\sqrt{\frac{1}{(\lambda y - C)^2}-1}\\
\pm\frac{dy}{\sqrt{\frac{1}{(\lambda y-C)^2}-1}} &= dx\\
\mp\frac{\sqrt{1-(\lambda y - C)^2}}{\lambda} &= x+D\\
\frac{1-(\lambda y - C)^2}{\lambda^2} &= (x+D)^2\\
\frac{1}{\lambda^2}-\left(y-\frac{C}{\lambda}\right)^2 &= (x+D)^2\\
(x+D)^2+\left(y-\frac{C}{\lambda}\right)^2 &= \frac{1}{\lambda^2}
\end{aligned}$$

As we see, this is an arc of a circle with center $(-D, C/\lambda)$ and radius $|1/\lambda|$.
Note: the constants C, D, λ can be determined from the information about the two boundary points and the constraint $\sigma = \int_{x_1}^{x_2} y\,dx = a$ (see Problem 9).

9. Given that the curve in Problem 8 passes through the points $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$ and the area beneath it and above the x axis is 1.1416, determine the unknown constants in the solution and hence obtain the specific solution. Also, plot the curve and find the length of the arc.
Answer: As we found in the answer of Problem 8, the curve is (an arc of) a circle and hence its equation is $(x-c_1)^2+(y-c_2)^2 = R^2$ with (c_1, c_2) being the center and R being the radius. So, instead of determining C, D, λ we determine c_1, c_2, R and hence obtain the specific solution. Now, from the symmetry of the circular segment $\big[$as can be seen from the points $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)\big]$ we know that the center of the circle is on the y axis and hence $c_1 = 0$. Thus, the equation of the circle becomes $x^2 + (y-c_2)^2 = R^2$. We also know from elementary geometry that the area σ of a segment of a circle with radius R is given by:

$$\sigma = \frac{1}{2}\left[2R^2\arcsin\left(\frac{l}{2R}\right) - l\sqrt{R^2-\left(\frac{l}{2}\right)^2}\right]$$

where l is the length of the chord (i.e. the straight base of the segment) which in our case is equal to $2\sqrt{2}$. On using a numerical solver (with $\sigma = 1.1416$ and $l = 2\sqrt{2}$) we find $R = 2$ and hence the equation of the circle becomes $x^2 + (y-c_2)^2 = 4$. We finally use one point $\big[$say $(\sqrt{2}, 0)\big]$ to find c_2, that is:

$$\left(\sqrt{2}\right)^2 + (0-c_2)^2 = 4 \qquad \text{and hence} \qquad c_2 = -\sqrt{2}$$

where we take the negative root for obvious geometric considerations. Therefore, the specific solution is $x^2 + \left(y+\sqrt{2}\right)^2 = 4$. The circle is plotted in Figure 31. Regarding the length of the arc, it is obvious that the central angle that subtends the arc $\big[$i.e. the angle between the points $(-\sqrt{2}, 0)$, $(0, -\sqrt{2})$ and $(\sqrt{2}, 0)\big]$ is a right angle and hence the arc is one quarter of the perimeter, i.e. it is π.

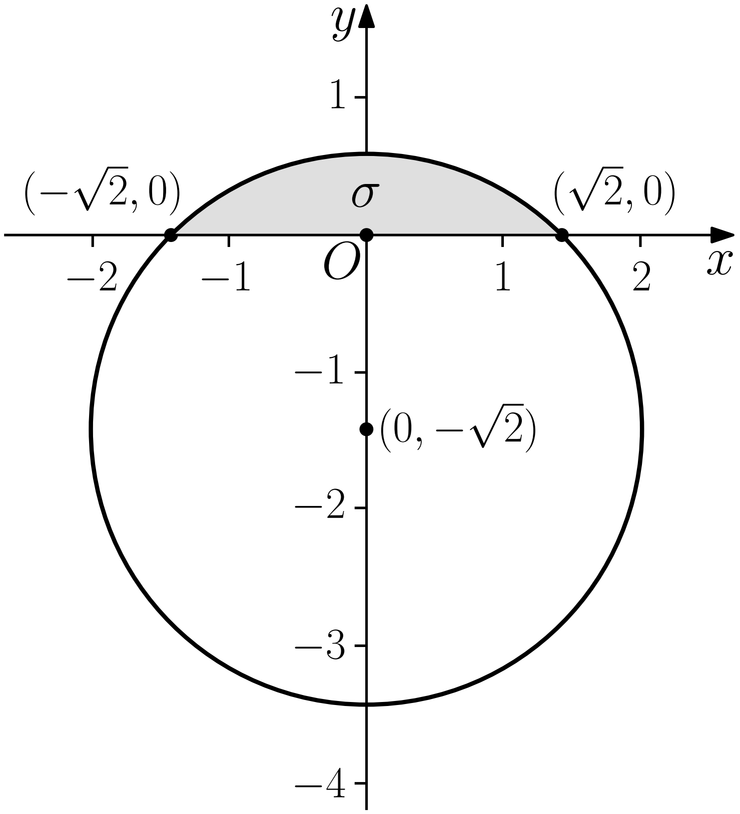

Figure 31: Plot of the circle $x^2+\left(y+\sqrt{2}\right)^2=4$ where the segment of area σ under the circular arc is shown in gray. See Problem 9 of § 2.4.

10. Given that the curve in Problem 8 passes through the points $(-1,5)$ and $(4,0)$ and the area beneath it and above the x axis is $\frac{25\pi}{4}$, determine the unknown constants in the solution and hence obtain the specific solution. Also, plot the curve and find the length of the arc.
Answer: As in Problem 9, the curve is a circle and its equation can be written as:

$$(x-c_1)^2+(y-c_2)^2=R^2 \tag{66}$$

where (c_1,c_2) is the center and R is the radius. Now, from the points $(-1,5)$ and $(4,0)$ we get:

$$\begin{aligned}(-1-c_1)^2+(5-c_2)^2 &= R^2 && (67)\\ (4-c_1)^2+(0-c_2)^2 &= R^2 && (68)\end{aligned}$$

On subtracting Eq. 68 from Eq. 67 we get:

$$\begin{aligned}(-1-c_1)^2+(5-c_2)^2-(4-c_1)^2-(0-c_2)^2 &= 0\\ 1+2c_1+c_1^2+25-10c_2+c_2^2-16+8c_1-c_1^2-c_2^2 &= 0\\ 10+10c_1-10c_2 &= 0\\ c_2 &= c_1+1 && (69)\end{aligned}$$

Also, on adding Eq. 68 to Eq. 67 we get:

$$(-1-c_1)^2+(5-c_2)^2+(4-c_1)^2+(0-c_2)^2 = 2R^2$$

$$
\begin{aligned}
1+2c_1+c_1^2+25-10c_2+c_2^2+16-8c_1+c_1^2+c_2^2 &= 2R^2 \\
21-3c_1+c_1^2-5c_2+c_2^2 &= R^2 \\
21-3c_1+c_1^2-5\left(c_1+1\right)+\left(c_1+1\right)^2 &= R^2 \qquad \text{(substituting from Eq. 69)} \\
21-3c_1+c_1^2-5c_1-5+c_1^2+2c_1+1 &= R^2 \\
2c_1^2-6c_1+17 &= R^2 \qquad (70)
\end{aligned}
$$

On solving Eq. 66 for y we get $y=\sqrt{R^2-\left(x-c_1\right)^2}+c_2$ where c_2 and R^2 are given by Eqs. 69 and 70 (in terms of c_1). Accordingly:

$$
\begin{aligned}
\sigma &= \int_{-1}^{+4} y\,dx \\
\frac{25\pi}{4} &= \int_{-1}^{+4}\left[\sqrt{R^2-\left(x-c_1\right)^2}+c_2\right]dx \\
\frac{25\pi}{4} &= \left[\frac{1}{2}R^2\arcsin\left(\frac{x-c_1}{R}\right)+\frac{1}{2}R\left(x-c_1\right)\sqrt{1-\left(\frac{x-c_1}{R}\right)^2}+c_2x\right]_{-1}^{+4} \\
\frac{25\pi}{4} &= \left[\frac{1}{2}R^2\arcsin\left(\frac{4-c_1}{R}\right)+\frac{1}{2}R\left(4-c_1\right)\sqrt{1-\left(\frac{4-c_1}{R}\right)^2}+c_2 4\right]- \\
&\qquad \left[\frac{1}{2}R^2\arcsin\left(\frac{-1-c_1}{R}\right)+\frac{1}{2}R\left(-1-c_1\right)\sqrt{1-\left(\frac{-1-c_1}{R}\right)^2}-c_2\right] \\
\frac{25\pi}{4} &= \frac{1}{2}\left(2c_1^2-6c_1+17\right)\arcsin\left(\frac{4-c_1}{\sqrt{2c_1^2-6c_1+17}}\right)+ \\
&\qquad \frac{1}{2}\sqrt{2c_1^2-6c_1+17}\left(4-c_1\right)\sqrt{1-\left(\frac{4-c_1}{\sqrt{2c_1^2-6c_1+17}}\right)^2}+4\left(c_1+1\right)- \\
&\qquad \frac{1}{2}\left(2c_1^2-6c_1+17\right)\arcsin\left(\frac{-1-c_1}{\sqrt{2c_1^2-6c_1+17}}\right)- \\
&\qquad \frac{1}{2}\sqrt{2c_1^2-6c_1+17}\left(-1-c_1\right)\sqrt{1-\left(\frac{-1-c_1}{\sqrt{2c_1^2-6c_1+17}}\right)^2}+\left(c_1+1\right)
\end{aligned}
$$

where in the last equation we substituted for c_2 and R from Eqs. 69 and 70. On using a numerical solver we find $c_1=-1$ and hence $c_2=0$ (from Eq. 69) and $R=5$ (from Eq. 70). Therefore, the specific solution is $\left(x+1\right)^2+y^2=25$. The circle is plotted in Figure 32. Regarding the length of the arc, it is obvious that the central angle that subtends the arc $\big[$i.e. the angle between the points $(-1,5)$, $(-1,0)$ and $(4,0)\big]$ is a right angle and hence the arc is one quarter of the perimeter, i.e. it is $\frac{5\pi}{2}$.

11. Find the shape of the closed plane curve of shortest length that encloses a given area.
Answer: If we repeat the analysis of Problem 6 then we have $H=\sqrt{1+y'^2}+\lambda\frac{1}{2}\left(xy'-y\right)$ where we shift the roles of F and G in that Problem (because we are trying to optimize the length while imposing the area constraint). Accordingly, the Euler-Lagrange equation (Eq. 22) for this Problem is:

$$
\frac{\partial}{\partial y}\left[\sqrt{1+y'^2}+\lambda\frac{1}{2}\left(xy'-y\right)\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[\sqrt{1+y'^2}+\lambda\frac{1}{2}\left(xy'-y\right)\right]\right) = 0
$$

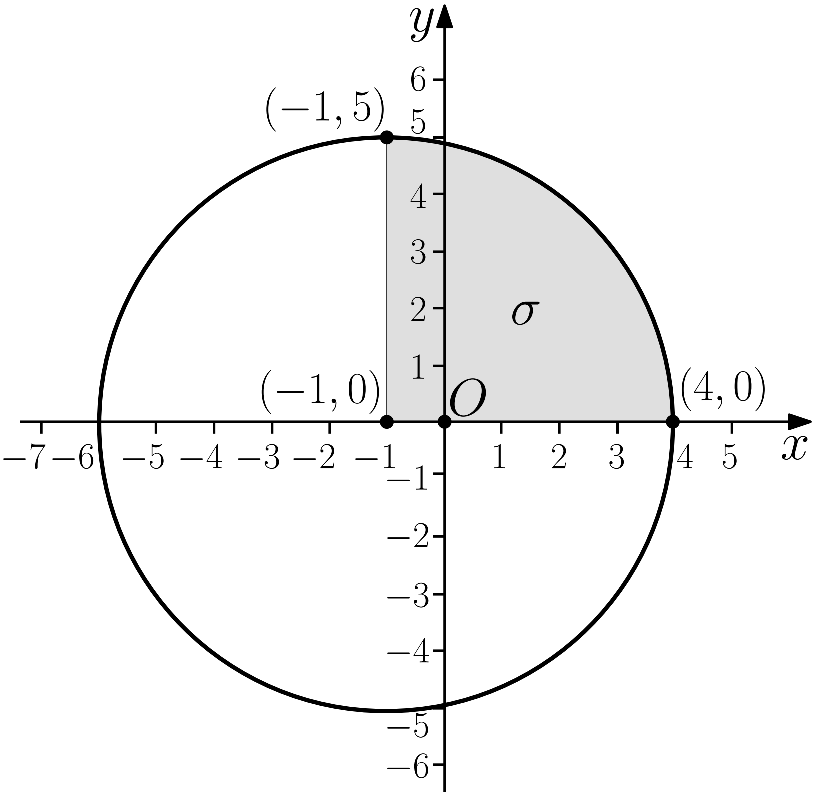

Figure 32: Plot of the circle $(x+1)^2 + y^2 = 25$ where the part of area σ under the circular arc is shown in gray. See Problem 10 of § 2.4.

$$
\begin{aligned}
-\frac{\lambda}{2} - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}} + \frac{\lambda x}{2}\right) &= 0 \\
\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}} + \frac{\lambda x}{2}\right) &= -\frac{\lambda}{2} \\
\frac{y'}{\sqrt{1+y'^2}} + \frac{\lambda x}{2} &= -\frac{\lambda x}{2} + C \\
\frac{y'}{\sqrt{1+y'^2}} &= -\lambda x + C \\
y'^2 &= (\lambda x - C)^2\left(1+y'^2\right) \\
y'^2 &= \frac{(\lambda x - C)^2}{1-(\lambda x - C)^2} \\
y' &= \pm\frac{(\lambda x - C)}{\sqrt{1-(\lambda x - C)^2}} \\
y &= \mp\frac{1}{\lambda}\sqrt{1-(\lambda x - C)^2} + D \\
(y-D)^2 &= \frac{1}{\lambda^2}\left[1-(\lambda x - C)^2\right]
\end{aligned}
$$

$$\left(x-\frac{C}{\lambda}\right)^2+\left(y-D\right)^2 = \frac{1}{\lambda^2}$$

This is an equation of a circle with center $(C/\lambda, D)$ and radius $|1/\lambda|$. So, the shape of the closed plane curve of shortest length that encloses a given area is a circle (noting that the solution cannot be a maximum because the length of the curve of a squeezed shape that encloses that area can diverge, e.g. a rectangle with one of its sides approaches zero).

Note: the result of this Problem is intuitive and can be concluded from Problem 6 as a corollary using the proof by contradiction method[60] because if the closed plane curve of a given enclosed area (say σ_0) and shortest length (say p_0 where p stands for perimeter) was not a circle then the circle of area σ_0 should have a longer perimeter than p_0 and hence the circle of perimeter p_0 should have smaller area than σ_0 and this contradicts the result of Problem 6 because the circle of perimeter p_0 should enclose maximum area.[61]

12. Find the equation of the closed plane curve of shortest length l that encloses an area $\sigma \simeq 28.2743$ and passes through the points $(6,-1)$ and $(3,2)$. Also, plot the result and find the length of the curve.

Answer: From the answer of Problem 11 we know that this curve is a circle, so we can write the equation of the curve as $(x-c_x)^2+(y-c_y)^2=R^2$ where (c_x, c_y) is the center of the circle and R is its radius. Now, $\sigma = 28.2743 = \pi R^2$ and hence $R=3$. Therefore, the equation of the curve becomes $(x-c_x)^2+(y-c_y)^2=9$. On substituting the points $(6,-1)$ and $(3,2)$ into this equation we get:

$$\begin{aligned}(6-c_x)^2+(-1-c_y)^2 &= 9 \quad (71)\\ (3-c_x)^2+(2-c_y)^2 &= 9 \quad (72)\end{aligned}$$

On subtracting Eq. 72 from Eq. 71 we get:

$$\begin{aligned}(6-c_x)^2+(-1-c_y)^2-(3-c_x)^2-(2-c_y)^2 &= 0\\ 36-12c_x+c_x^2+1+2c_y+c_y^2-9+6c_x-c_x^2-4+4c_y-c_y^2 &= 0\\ 24-6c_x+6c_y &= 0\\ c_y &= c_x-4 \quad (73)\end{aligned}$$

On substituting from this equation into Eq. 72 we get:

$$\begin{aligned}(3-c_x)^2+(6-c_x)^2 &= 9\\ 9-6c_x+c_x^2+36-12c_x+c_x^2 &= 9\\ c_x^2-9c_x+18 &= 0\\ (c_x-3)(c_x-6) &= 0\end{aligned}$$

Accordingly, $c_x=3$ or $c_x=6$ and hence $c_y=-1$ or $c_y=2$ (from Eq. 73). Hence, the equation of the circle is either $(x-3)^2+(y+1)^2=9$ or $(x-6)^2+(y-2)^2=9$ (i.e. we actually have two circles that satisfy our requirements). The result is plotted in Figure 33. The length of the curve (i.e. the circumference of the circle) is $l=2\pi R=6\pi$.

---

[60] The proof by contradiction method should be linked in many cases to the "Principle of Reciprocity" which states (in one of its forms) that if y optimizes $I_1[y]$ subject to the condition that $I_2[y]$ is constant then y optimizes (usually in opposite sense) $I_2[y]$ subject to the condition that $I_1[y]$ is constant. However, this principle is not rigorous and hence it may be violated in some cases. We note that the use of $I_1[y]$ and $I_2[y]$ is a reference to the Lagrange multipliers formulation and symbolism (see § 1.8).

[61] For circle, $\sigma=\pi r^2$ and $p=2\pi r$ (where σ is its area, p is its perimeter, and r is its radius) which can be combined to obtain $\sigma=\frac{p^2}{4\pi}$ and hence the area increases/decreases as the perimeter increases/decreases (and vice versa).

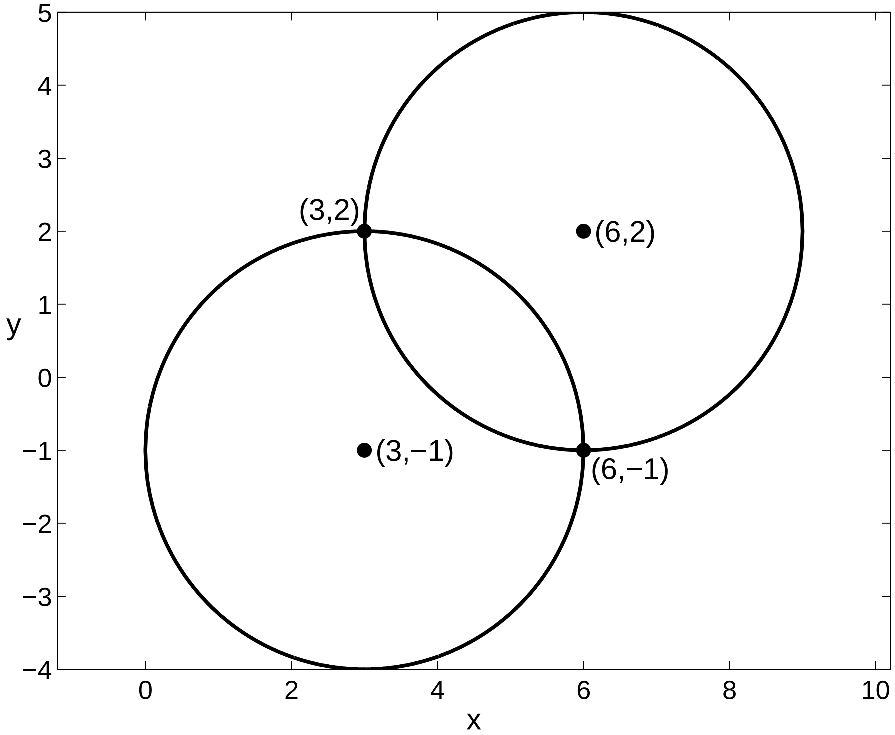

Figure 33: Plot of the circles $(x-3)^2+(y+1)^2=9$ and $(x-6)^2+(y-2)^2=9$ of Problem 12 of § 2.4.

Chapter 3
Optimal Surfaces

In this chapter we present and solve problems about topics and applications of the mathematics of variation related to optimal surfaces, i.e. we are looking in these problems to certain surfaces (or 2D objects) that optimize something (such as area).

3.1 2D Planar Shapes of Optimal Perimeter

From the title of this section, it is clear that in this type of problems it is required to optimize the perimeter of a given 2D planar shape (such as triangle or rectangle) subject to certain condition(s) such as being of fixed area. We note that due to the simplicity of this type of problems we use ordinary calculus (occasionally in association with the Lagrange multipliers technique) with no need for the variational formulation of the Euler-Lagrange equation.

Problems

1. What is the shape of the rectangle of fixed area and optimal perimeter?[62]
Answer: If p is the perimeter of the rectangle, σ is its area, L is its length and W is its width then we have $p = 2(L+W)$ and $\sigma = LW$. Therefore, $W = \frac{\sigma}{L}$ and hence $p = 2\left(L + \frac{\sigma}{L}\right)$. On differentiating p with respect to L and setting the derivative to zero (since it vanishes when p is optimal) we obtain the optimization condition, that is:
$$\frac{dp}{dL} = 2\left(1 - \frac{\sigma}{L^2}\right) = 0 \qquad \text{and hence} \qquad L = \sqrt{\sigma} \quad \text{and} \quad W = \frac{\sigma}{L} = \frac{\sigma}{\sqrt{\sigma}} = \sqrt{\sigma}$$
Therefore, $L = W = \sqrt{\sigma}$ which means that the rectangle should be a square to optimize the perimeter.
Note: it is obvious that the optimal perimeter in this Problem is a minimum (not a maximum) because the perimeter can diverge when the width of the rectangle approaches zero (noting that the area is fixed).
2. Re-solve Problem 1 but this time use the Lagrange multipliers technique (see § 1.8).
Answer: Let the length and width of the rectangle be L and W and hence we want to optimize $f = 2(L+W)$ (which is the perimeter) subject to the constraint that the area LW is constant (and hence $g = LW$ is constant). So, we should optimize $h = f + \lambda g = 2(L+W) + \lambda LW$ by taking the partial derivatives of h with respect to the variables L and W and setting the derivatives to zero to obtain the optimization conditions, that is:
$$\begin{aligned} \frac{\partial h}{\partial L} &= 2 + \lambda W = 0 \qquad \text{and hence} \qquad W = -2/\lambda \\ \frac{\partial h}{\partial W} &= 2 + \lambda L = 0 \qquad \text{and hence} \qquad L = -2/\lambda \end{aligned}$$
Accordingly, $L = W$ which again means that the rectangle should be a square to optimize its perimeter.
Note: again, the optimal perimeter in this Problem is a minimum (for the same reason).
3. What is the shape of the triangle with fixed area and optimal perimeter?[63]
Answer: Referring to Figure 34 we have:

---

[62] "Shape" in this context refers to a distinctive feature in the generic shape (which is rectangle) and hence this question can be posed as: what is the length to width ratio of the rectangle of fixed area and optimal perimeter?

[63] As indicated earlier, "shape" in questions like this means a distinctive feature in the generic shape of the object (such as being isosceles or equilateral or having certain height to base ratio).

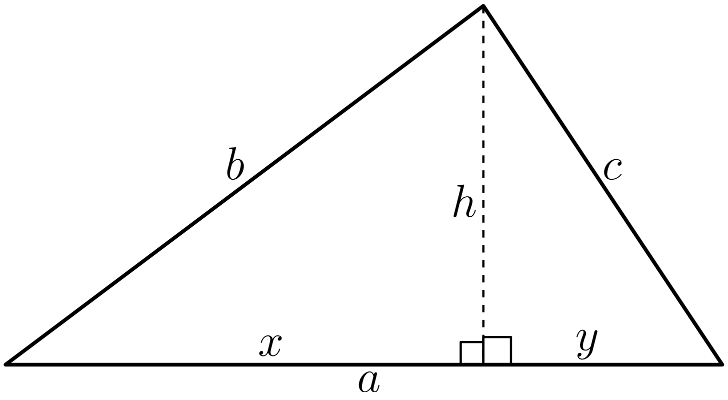

Figure 34: A schematic illustration of the setting of Problem 3 of § 3.1 where a triangle of fixed area σ is to be optimized in its perimeter $p = a + b + c$.

$$\begin{aligned} p &= a+b+c = (x+y) + \sqrt{x^2+h^2} + \sqrt{y^2+h^2} \\ \sigma &= \frac{1}{2}ah = \frac{h}{2}(x+y) \end{aligned}$$

where p is the perimeter, a, b, c are the lengths of the triangle sides, σ is its area, h is its height, and $x + y = a$. Now, we are supposed to optimize p subject to the restriction on σ (since it is fixed), and hence we use the Lagrange multipliers method (see § 1.8) with $f + \lambda g = p + \lambda\sigma$, that is:

$$p + \lambda\sigma = (x+y) + \sqrt{x^2+h^2} + \sqrt{y^2+h^2} + \lambda\frac{h}{2}(x+y)$$

To optimize $p+\lambda\sigma$ we need to take the partial derivatives of $p+\lambda\sigma$ with respect to the variables x, y, h and set the derivatives to zero to obtain the optimization conditions, that is:

$$\frac{\partial}{\partial x}(p+\lambda\sigma) = 1 + \frac{x}{\sqrt{x^2+h^2}} + \frac{\lambda h}{2} = 0 \quad (74)$$

$$\frac{\partial}{\partial y}(p+\lambda\sigma) = 1 + \frac{y}{\sqrt{y^2+h^2}} + \frac{\lambda h}{2} = 0 \quad (75)$$

$$\frac{\partial}{\partial h}(p+\lambda\sigma) = \frac{h}{\sqrt{x^2+h^2}} + \frac{h}{\sqrt{y^2+h^2}} + \frac{\lambda}{2}(x+y) = 0 \quad (76)$$

On subtracting Eq. 75 from Eq. 74 we get:

$$\begin{aligned} \frac{x}{\sqrt{x^2+h^2}} - \frac{y}{\sqrt{y^2+h^2}} &= 0 \\ \frac{x}{\sqrt{x^2+h^2}} &= \frac{y}{\sqrt{y^2+h^2}} \\ \frac{x^2}{x^2+h^2} &= \frac{y^2}{y^2+h^2} \\ x^2\left(y^2+h^2\right) &= y^2\left(x^2+h^2\right) \\ x^2y^2 + x^2h^2 - x^2y^2 - y^2h^2 &= 0 \\ h^2\left(x^2-y^2\right) &= 0 \\ x^2 - y^2 &= 0 \qquad (h \neq 0) \\ (x-y)(x+y) &= 0 \\ x - y &= 0 \qquad (x+y = a \neq 0) \end{aligned}$$

$$x = y$$

Therefore, the area becomes:

$$\sigma = \frac{h}{2}(x+y) = \frac{h}{2}(x+x) = \frac{h}{2}(2x) = xh \tag{77}$$

while Eqs. 74-76 reduce to the following two equations:

$$1 + \frac{x}{\sqrt{x^2+h^2}} + \frac{\lambda h}{2} = 0 \tag{78}$$

$$\frac{2h}{\sqrt{x^2+h^2}} + \lambda x = 0 \tag{79}$$

On multiplying Eq. 78 with x and Eq. 79 with $h/2$ and subtracting the second from the first we get:

$$\begin{aligned}
x + \frac{x^2}{\sqrt{x^2+h^2}} - \frac{h^2}{\sqrt{x^2+h^2}} &= 0 \\
x &= \frac{h^2 - x^2}{\sqrt{x^2+h^2}} \\
x^2 &= \frac{\left(h^2 - x^2\right)^2}{x^2+h^2} \\
x^4 + x^2h^2 &= h^4 - 2x^2h^2 + x^4 \\
x^2h^2 &= h^4 - 2x^2h^2 \\
\sigma^2 &= h^4 - 2\sigma^2 && \text{(substituting from Eq. 77)} \\
3\sigma^2 &= h^4 \\
\sigma &= \frac{h^2}{\sqrt{3}} \\
x &= \frac{h}{\sqrt{3}} && \text{(using Eq. 77)}
\end{aligned}$$

Accordingly, the length of the three sides are:

$$\begin{aligned}
a &= x + y = 2x = \frac{2h}{\sqrt{3}} \\
b &= \sqrt{x^2+h^2} = \sqrt{\frac{h^2}{3} + h^2} = \sqrt{\frac{4h^2}{3}} = \frac{2h}{\sqrt{3}} && \text{(Using Pythagoras)} \\
c &= \sqrt{y^2+h^2} = \sqrt{x^2+h^2} = \sqrt{\frac{h^2}{3} + h^2} = \sqrt{\frac{4h^2}{3}} = \frac{2h}{\sqrt{3}} && \text{(Using Pythagoras)}
\end{aligned}$$

So, $a = b = c = \frac{2h}{\sqrt{3}}$ which means that our triangle is equilateral.
Note 1: it should be obvious that the optimal perimeter in this Problem is a minimum (not a maximum) because for a triangle with a fixed area if the height (or the base) approaches zero the perimeter diverges.
Note 2: although the above solution is based on the configuration of Figure 34 (which seems rather restricted) it can be easily adapted to include all possible configurations.

4. What is the shape of the ellipse with fixed area and optimal perimeter (or circumference)?
   Answer: The perimeter of ellipse is:

$$p = \int_0^{\pi/2} 4a\sqrt{1 - e^2 \sin^2\theta}\, d\theta$$

where a is its semi-major axis while e is its eccentricity. Now, we are supposed to optimize p subject to the restriction on σ (since it is fixed), and hence we can use the Lagrange multipliers method with $f + \lambda g = p + \lambda\sigma$, that is:

$$p + \lambda\sigma = \int_0^{\pi/2} 4a\sqrt{1 - e^2 \sin^2\theta}\, d\theta + \lambda\sigma$$

So, we need to optimize $p + \lambda\sigma$ by taking its derivative with respect to e (since $p + \lambda\sigma$ is a function of e) and setting the derivative to zero to obtain the optimization condition, that is:

$$\begin{aligned} \frac{d}{de}\left(p + \lambda\sigma\right) &= 0 \\ \int_0^{\pi/2} \frac{\partial}{\partial e}\left(4a\sqrt{1 - e^2 \sin^2\theta}\right) d\theta + 0 &= 0 \\ \int_0^{\pi/2} \frac{-4ae\sin^2\theta}{\sqrt{1 - e^2 \sin^2\theta}}\, d\theta &= 0 \end{aligned}$$

As we see, this integral can be zero only if $e = 0$ which means that the ellipse is a circle. So, the ellipse of fixed area and optimal perimeter is a circle (which is a special case of ellipse corresponding to $e = 0$).

Note 1: in differentiating the integral we use the fact that if ξ is a function of two variables (say t and x) and

$$\phi\left(t\right) = \int_\alpha^\beta \xi\left(t, x\right)\, dx$$

(with α and β being constants) then

$$\frac{d\phi(t)}{dt} = \int_\alpha^\beta \frac{\partial\xi\left(t, x\right)}{\partial t}\, dx$$

Note 2: the optimal perimeter in this Problem is obviously a minimum (not a maximum) because for an ellipse of fixed area the perimeter diverges when the minor axis approaches zero (i.e. e approaches one).

3.2 2D Planar Shapes of Optimal Area

From the title of this section, it is obvious that in this type of problems it is required to optimize the area of a given 2D planar shape (such as triangle or rectangle) subject to certain condition(s) such as being of fixed perimeter. As in the Problems of § 3.1, we use in solving these problems ordinary calculus only with no need for using the Euler-Lagrange variational formulation.

Problems

1. What is the shape of the rectangle of fixed perimeter p and optimal area?

   Answer: If we use the setting and labeling of Problem 1 of § 3.1 and follow a similar reasoning then we can say: in the present Problem we have $\sigma = LW$ and $p = 2\left(L + W\right)$ and therefore $W = (p/2) - L$ and hence $\sigma = L\left[(p/2) - L\right]$. On differentiating σ with respect to L and setting the derivative to zero (since it vanishes when σ is optimal) we obtain the optimization condition, that is:

$$\frac{d\sigma}{dL} = \frac{p}{2} - 2L = 0 \qquad \text{and hence} \qquad L = \frac{p}{4} \quad \text{and} \quad W = \frac{p}{2} - L = \frac{p}{2} - \frac{p}{4} = \frac{p}{4}$$

   Therefore, $L = W = p/4$ which means that the rectangle should be a square to optimize its area.

   Note: it is obvious that the optimal area in this Problem is a maximum (not a minimum) because the area can approach zero when the width of the rectangle approaches zero (noting that the perimeter is fixed).

2. Re-solve Problem 1 but this time use the Lagrange multipliers technique (see § 1.8).
   Answer: If we use the setting and labeling of Problem 2 of § 3.1 and follow a similar reasoning then we can say: in the present Problem we want to optimize $f = LW$ (which is the area) subject to the constraint that the perimeter $2(L+W)$ is constant (and hence $g = L+W$ is constant). So, we should optimize $h = f + \lambda g = LW + \lambda(L+W)$ by taking the partial derivative of h with respect to the variables L and W and setting the derivatives to zero to obtain the optimization conditions, that is:
   $$\begin{aligned} \frac{\partial h}{\partial L} &= W + \lambda = 0 \qquad \text{and hence} \qquad W = -\lambda \\ \frac{\partial h}{\partial W} &= L\ + \lambda = 0 \qquad \text{and hence} \qquad L\ = -\lambda \end{aligned}$$
   Accordingly, $L = W$ which again means that the rectangle should be a square to optimize its area.
   Note: again, the optimal area in this Problem is a maximum (for the same reason).
3. Re-solve Problem 1 but this time use the proof by contradiction method.
   Answer: The result of Problem 1 (or Problem 2) can be obtained as a corollary from the result of Problem 1 of § 3.1 (or Problem 2 of § 3.1) because if the rectangle of the given perimeter (say p_0) and maximum area (say σ_0) was not a square then the square of perimeter p_0 should have a smaller area than σ_0 and hence the square of area σ_0 should have longer perimeter than p_0 and this contradicts the result of Problem 1 of § 3.1 (or Problem 2 of § 3.1) because the square of area σ_0 should have minimum perimeter.[64]
   Note: the course of action in the present Problem can be reversed by using the result of Problem 1 of the present section (or Problem 2 of the present section) in conjunction with the proof by contradiction method to establish the result of Problem 1 of § 3.1 (or Problem 2 of § 3.1). In this case we would say: the result of Problem 1 of § 3.1 (or Problem 2 of § 3.1) can be obtained as a corollary from the result of Problem 1 of the present section (or Problem 2 of the present section) because if the rectangle of a given area (say σ_0) and minimum perimeter (say p_0) was not a square then the square of area σ_0 should have a longer perimeter than p_0 and hence the square of perimeter p_0 should have smaller area than σ_0 and this contradicts the result of Problem 1 of the present section (or Problem 2 of the present section) because the square of perimeter p_0 should have maximum area.[65]
4. What is the shape of the triangle with fixed perimeter and optimal area?
   Answer: It is equilateral. This is a corollary of the result of Problem 3 of § 3.1 because if the equilateral triangle with fixed perimeter (say p_0) is not optimal (i.e. maximum; see the upcoming note) in area then there should be a non-equilateral triangle with perimeter p_0 and with maximum area (say σ_0) and hence an equilateral triangle of area σ_0 will have a longer perimeter than p_0 which contradicts the result of Problem 3 of § 3.1 because the equilateral triangle of area σ_0 has minimum perimeter.[66]
   Note: it should be obvious that the optimal area in this Problem is a maximum (not a minimum) because for a triangle with a fixed perimeter if the height (or the base) approaches zero the area converges to zero.
5. What is the shape of the triangle of optimal area whose perimeter and one of its sides are fixed (i.e. only two of its sides can vary while keeping their sum fixed).[67]
   Answer: If we follow the method of Problem 3 of § 3.1 (assuming the fixed side to be a) then this

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[64] For square, $p = 4x$ and $\sigma = x^2$ (where p is its perimeter, σ is its area, and x is the length of its sides) which can be combined to obtain $p = 4\sqrt{\sigma}$ and hence the perimeter increases/decreases as the area increases/decreases (and vice versa).

[65] For square, $\sigma = x^2$ and $p = 4x$ (where σ is its area, p is its perimeter, and x is the length of its sides) which can be combined to obtain $\sigma = p^2/16$ and hence the area increases/decreases as the perimeter increases/decreases (and vice versa).

[66] For equilateral triangle, $p = 3a$ and $\sigma = \frac{\sqrt{3}}{4}a^2$ (where p is its perimeter, σ is its area, and a is the length of its sides) which can be combined to obtain $p = 6\sqrt{\frac{\sigma}{\sqrt{3}}}$ and hence the perimeter increases/decreases as the area increases/decreases (and vice versa).

[67] We can also characterize this triangle as: the length of one of its sides and the sum of the lengths of its two other sides are fixed.

time we have $f + \lambda g = \sigma + \lambda p = \frac{h}{2}(x+y) + \lambda(x+y) + \lambda\sqrt{x^2+h^2} + \lambda\sqrt{y^2+h^2}$ (because here we want to optimize the area subject to a perimeter constraint) and hence:

$$\begin{aligned}\frac{\partial}{\partial x}(\sigma + \lambda p) &= \frac{h}{2} + \lambda\left(1 + \frac{x}{\sqrt{x^2+h^2}}\right) = 0\\ \frac{\partial}{\partial y}(\sigma + \lambda p) &= \frac{h}{2} + \lambda\left(1 + \frac{y}{\sqrt{y^2+h^2}}\right) = 0\end{aligned}$$

On subtracting the second equation from the first equation we get:

$$\frac{x}{\sqrt{x^2+h^2}} - \frac{y}{\sqrt{y^2+h^2}} = 0$$

and hence $x = y$ (as shown in Problem 3 of § 3.1) which means that the triangle is an isosceles.
Note 1: it should be obvious that the optimal area in this Problem is a maximum (not a minimum) because for a triangle with a fixed perimeter (regardless of having a fixed side or not) if the height approaches zero the area converges to zero.
Note 2: if the length of the fixed side is half the sum of the other two sides then the optimal triangle of this Problem becomes equilateral and this Problem becomes an instance of Problem 4.

6. What is the shape of the ellipse with fixed perimeter and optimal area?
Answer: It should be a circle. This is a corollary of the result of Problem 4 of § 3.1 because if the circle with the fixed perimeter (say p_0) is not optimal (i.e. maximum; see the upcoming note) in area then there should be an ellipse with perimeter p_0 and with maximum area (say σ_0) and hence a circle of area σ_0 will have a longer perimeter than p_0 which contradicts the result of Problem 4 of § 3.1 because the circle of area σ_0 has minimum perimeter.
Note: it should be obvious that the optimal area in this Problem is a maximum (not a minimum) because for an ellipse with a fixed perimeter if the minor axis approaches zero (i.e. e approaches one) the area converges to zero.

3.3 2D Planar Shapes Inside Other 2D Planar Shapes

In this type of problems it is required to optimize a certain property (such as area) of a given 2D planar shape (such as triangle) that is confined inside another 2D planar shape (such as circle). Again, we use in solving these problems ordinary calculus only with no need for the variational formulation of the Euler-Lagrange equation.

Problems

1. What is the shape of the triangle of optimal perimeter inscribed inside a circle?
Answer: Referring to Figure 35, we use a Cartesian coordinate system centered on the center of the circle. With no loss of generality, we put one of the triangle vertices (i.e. vertex A) on the positive x axis. Now, because the size of the radius of the circle does not affect the basic shape of the inscribed triangle (since the radius is just a scaling factor that scales the triangle up or down without affecting its features) we use (with no loss of generality) a unit circle. This is to simplify and reduce the required algebra. To formulate the Problem variationally we introduce two positive angles: θ (which is the positive angle AOB) and ϕ (which is the positive angle AOC). So, the setting of the Problem is as in Figure 35.
Now, from Figure 35 the perimeter p of the triangle is given by:

$$\begin{aligned}p &= a + c + b\\ &= \sqrt{(1-\cos\theta)^2 + \sin^2\theta} + \sqrt{(1-\cos\phi)^2 + \sin^2\phi} + \sqrt{(\cos\theta - \cos\phi)^2 + (\sin\theta - \sin\phi)^2}\\ &= \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} + \sqrt{1 - 2\cos\phi + \cos^2\phi + \sin^2\phi} +\end{aligned}$$

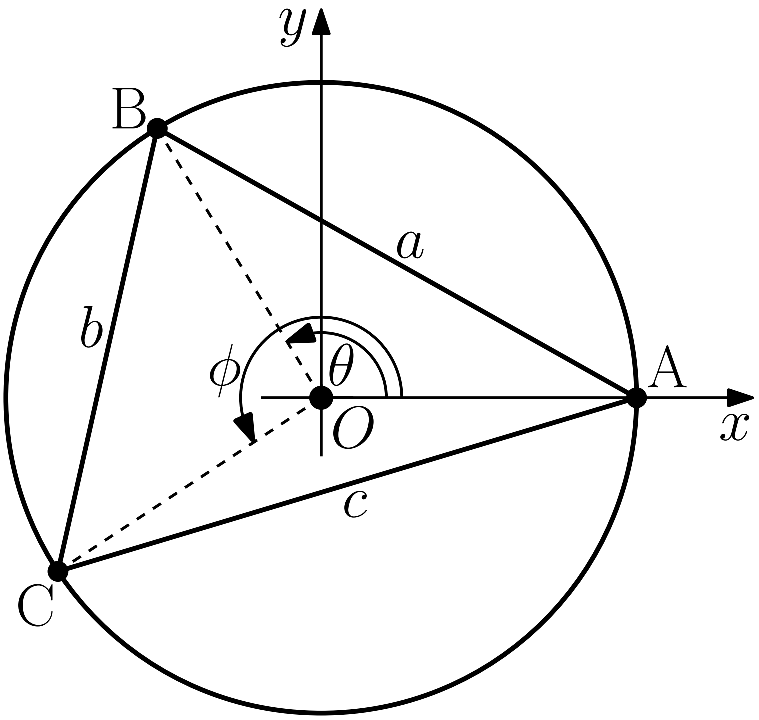

Figure 35: The setting of Problem 1 of § 3.3 where a triangle with vertices A$(1,0)$, B$(\cos\theta, \sin\theta)$ and C$(\cos\phi, \sin\phi)$ is inscribed inside a unit circle centered on the origin of coordinates with a, b, c being the lengths of the sides of the triangle.

$$\begin{aligned}
& \sqrt{\cos^2\theta - 2\cos\theta\cos\phi + \cos^2\phi + \sin^2\theta - 2\sin\theta\sin\phi + \sin^2\phi} \\
= \;& \sqrt{2 - 2\cos\theta} + \sqrt{2 - 2\cos\phi} + \sqrt{2 - 2\cos\theta\cos\phi - 2\sin\theta\sin\phi} \\
= \;& \sqrt{2}\left[\sqrt{1-\cos\theta} + \sqrt{1-\cos\phi} + \sqrt{1 - \cos\theta\cos\phi - \sin\theta\sin\phi}\right] \\
= \;& \sqrt{2}\left[\sqrt{1-\cos\theta} + \sqrt{1-\cos\phi} + \sqrt{1 - \cos(\theta-\phi)}\right]
\end{aligned}$$

To optimize p we take its partial derivatives with respect to the variables θ and ϕ and set the derivatives to zero to obtain the optimization conditions, that is:

$$\begin{aligned}
\frac{1}{\sqrt{2}}\frac{\partial p}{\partial\theta} &= 0 \\
\frac{\sin\theta}{2\sqrt{1-\cos\theta}} + \frac{\sin(\theta-\phi)}{2\sqrt{1-\cos(\theta-\phi)}} &= 0 \\
\frac{\sin\theta}{\sqrt{1-\cos\theta}} &= -\frac{\sin(\theta-\phi)}{\sqrt{1-\cos(\theta-\phi)}} \\
\frac{\sin^2\theta}{1-\cos\theta} &= \frac{\sin^2(\theta-\phi)}{1-\cos(\theta-\phi)} \\
\frac{1-\cos^2\theta}{1-\cos\theta} &= \frac{1-\cos^2(\theta-\phi)}{1-\cos(\theta-\phi)} \\
1+\cos\theta &= 1+\cos(\theta-\phi) \\
\cos\theta &= \cos(\theta-\phi) \qquad (80)
\end{aligned}$$

AND

$$\begin{aligned}
\frac{1}{\sqrt{2}}\frac{\partial p}{\partial \phi} &= 0 \\
\frac{\sin\phi}{2\sqrt{1-\cos\phi}} - \frac{\sin(\theta-\phi)}{2\sqrt{1-\cos(\theta-\phi)}} &= 0 \\
\frac{\sin\phi}{\sqrt{1-\cos\phi}} &= \frac{\sin(\theta-\phi)}{\sqrt{1-\cos(\theta-\phi)}} \\
\frac{\sin^2\phi}{1-\cos\phi} &= \frac{\sin^2(\theta-\phi)}{1-\cos(\theta-\phi)} \\
\frac{1-\cos^2\phi}{1-\cos\phi} &= \frac{1-\cos^2(\theta-\phi)}{1-\cos(\theta-\phi)} \\
1+\cos\phi &= 1+\cos(\theta-\phi) \\
\cos\phi &= \cos(\theta-\phi) \qquad (81)
\end{aligned}$$

On combining Eqs. 80 and 81 we get $\cos\theta = \cos\phi = \cos(\theta-\phi)$. Now, if we note that $\cos\phi = \cos(2\pi-\phi)$ and $\cos(\theta-\phi) = \cos(\phi-\theta)$ then we have $\cos\theta = \cos(2\pi-\phi) = \cos(\phi-\theta)$. Noting that θ is the positive angle AOB, $(2\pi-\phi)$ is the positive angle COA, and $(\phi-\theta)$ is the positive angle BOC, we conclude (within the given constraints) that the angles AOB, BOC and COA are equal which means that the triangle is equilateral.
Note: it should be obvious that the optimal perimeter in this Problem is a maximum (not a minimum) because the perimeter can converge to zero when the vertices of the triangle become too close to each other.[68]

2. Calculate the length of the longest perimeter of a triangle inscribed in a circle of radius R.
Answer: From the result of Problem 1, the inscribed triangle of longest perimeter is equilateral and hence the length is:

$$p = 6R\cos\frac{\pi}{6} = 6R \times \frac{\sqrt{3}}{2} = 3\sqrt{3}R$$

3. What is the shape of the triangle of optimal area inscribed inside a circle?
Answer: We refer to Figure 36 where the setting of the Problem is illustrated. Because the area σ of an inscribed triangle is equal to the area of the circle minus the sum S of the areas of the segments (shaded gray in Figure 36) we will optimize (i.e. minimize) S rather than optimize (i.e. maximize) σ. Now, the area σ_s of a segment of a circle of radius r is $\sigma_s = \frac{r^2}{2}(\alpha - \sin\alpha)$ where α is the subtended angle. With no loss of generality we can use a circle of radius $r = \sqrt{2}$ (because the radius r is just a scaling factor that does not affect the basic shape of the inscribed triangle) and hence the formula of σ_s becomes $\sigma_s = \alpha - \sin\alpha$. So, our task is to optimize S which is the sum of $\sigma_1, \sigma_2, \sigma_3$ (see Figure 36) that is:

$$\begin{aligned}
S &= \sigma_1 + \sigma_2 + \sigma_3 \\
&= \big[\theta - \sin\theta\big] + \big[\phi - \sin\phi\big] + \big[(2\pi-\theta-\phi) - \sin(2\pi-\theta-\phi)\big]
\end{aligned}$$

Now, to optimize S we take its partial derivatives with respect to θ and ϕ and set the derivatives to zero to obtain the optimization conditions, that is:

$$\begin{aligned}
\frac{\partial S}{\partial \theta} &= 1 - \cos\theta - 1 + \cos(2\pi-\theta-\phi) = -\cos\theta + \cos(2\pi-\theta-\phi) = 0 \qquad (82) \\
\frac{\partial S}{\partial \phi} &= 1 - \cos\phi - 1 + \cos(2\pi-\theta-\phi) = -\cos\phi + \cos(2\pi-\theta-\phi) = 0 \qquad (83)
\end{aligned}$$

On subtracting Eq. 82 from Eq. 83 we get:

$$\cos\theta - \cos\phi = 0$$

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[68] In fact, the perimeter of the inscribed equilateral triangle (which is $p = 3\sqrt{3}R$ with R being the radius of the circle) is greater than the perimeter in another limiting case when two vertices approach each other while the other vertex is on the other side of the circle (and hence the perimeter approaches $p = 4R$).

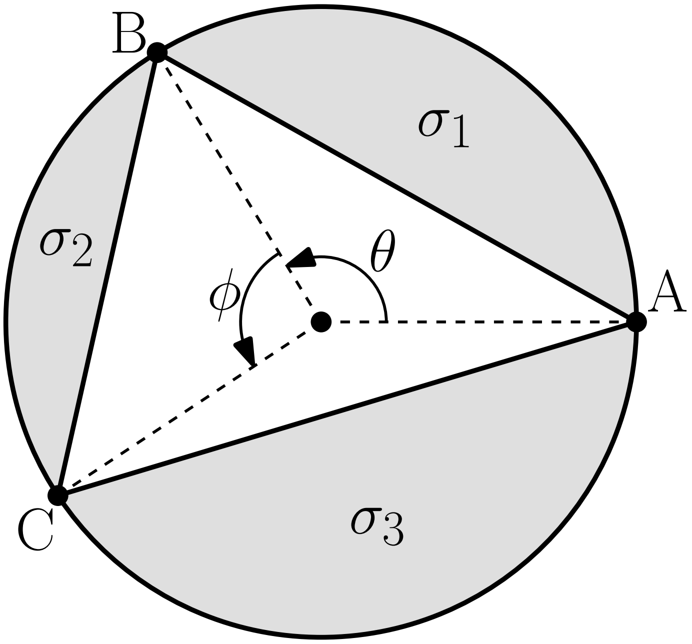

Figure 36: The setting of Problem 3 of § 3.3 where a triangle with vertices A, B and C is inscribed inside a circle of radius $r = \sqrt{2}$.

$$\cos\theta = \cos\phi \tag{84}$$

Also, On adding Eq. 82 and Eq. 83 we get:

$$\begin{aligned} -\cos\theta - \cos\phi + 2\cos(2\pi - \theta - \phi) &= 0 \\ -2\cos\theta + 2\cos(2\pi - \theta - \phi) &= 0 \qquad \text{(substituting from Eq. 84)} \\ \cos\theta &= \cos(2\pi - \theta - \phi) \qquad (85) \end{aligned}$$

So, from Eqs. 84 and 85 we can see that (within the given restrictions) $\theta = \phi = 2\pi - \theta - \phi$ which means that the three angles that determine the shape of the triangle are equal and hence the triangle is equilateral.

Note 1: it is obvious that in this Problem the optimal area of the triangle is a maximum (not a minimum) because the area can converge to zero when the vertices of the triangle (or two of them) become too close to each other.

Note 2: in our answer we optimized the sum S of the circle segments $\sigma_1, \sigma_2, \sigma_3$ instead of optimizing the area of the triangle directly (by optimizing the sum of the three inner triangles seen in Figure 36) to avoid necessary (and rather complicated) justifications according to some possible settings and configurations of the problem.

4. Calculate the area of the triangle of maximum area inscribed in a circle of radius R.
Answer: From the result of Problem 3, the inscribed triangle of maximum area is equilateral and hence the area is:

$$\sigma = \frac{1}{2} \times \sqrt{3}R \times \sqrt{3}R \sin\frac{\pi}{3} = \frac{1}{2} \times \sqrt{3}R \times \sqrt{3}R\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}R^2$$

5. What is the shape of the rectangle of optimal perimeter inscribed inside a circle?
Answer: We use a Cartesian coordinate system centered on the center of the circle with the sides of the rectangle being parallel to the axes (see Figure 37). Because the size of the radius of the circle does not affect the basic shape of the inscribed rectangle (since the radius is just a scale factor) we use (with

no loss of generality) a unit circle. Due to the symmetry, the perimeter p is given by $p = 4(\alpha+\beta)$ where α and β are the coordinates of the vertex in the first quadrant. Hence, we are required to optimize $\alpha + \beta$ (because 4 is no more than a scale factor). Now, the vertex (α, β) is on the circle and hence it should satisfy the equation of the circle $x^2 + y^2 = 1$, that is:

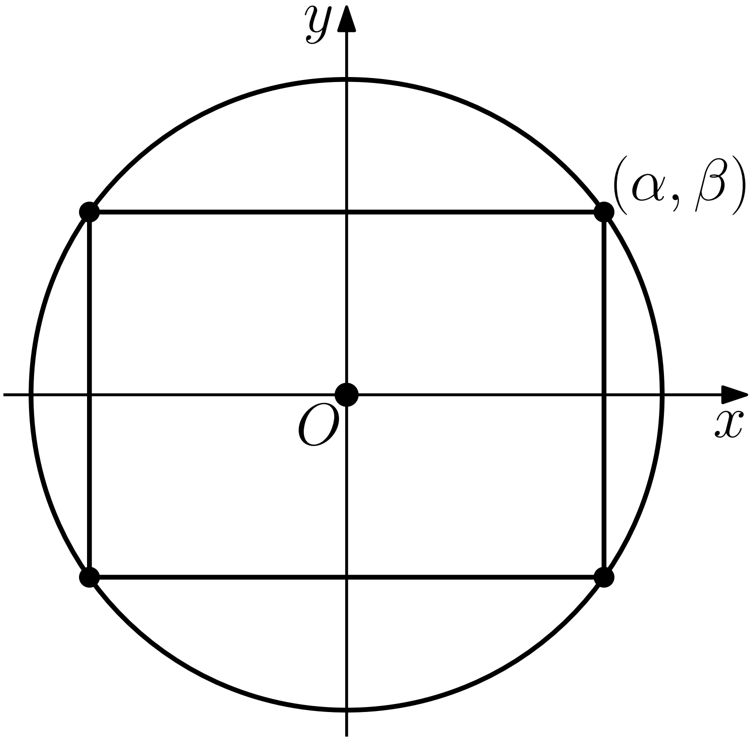

Figure 37: The setting of Problem 5 of § 3.3 where a rectangle (with its sides being parallel to the coordinate axes) is inscribed inside a unit circle centered on the origin of coordinates with α and β being the x and y coordinates of the vertex in the first quadrant.

$$\begin{aligned} \alpha^2 + \beta^2 &= 1 \\ \beta &= \sqrt{1 - \alpha^2} \end{aligned}$$

Accordingly, we are required to optimize $\alpha + \sqrt{1 - \alpha^2}$ by taking its derivative with respect to α and setting it to zero to obtain the optimization condition, that is:

$$\begin{aligned} \frac{d}{d\alpha}\left(\alpha + \sqrt{1 - \alpha^2}\right) &= 0 \\ 1 + \frac{-\alpha}{\sqrt{1 - \alpha^2}} &= 0 \\ \frac{\alpha^2}{1 - \alpha^2} &= 1 \\ 2\alpha^2 &= 1 \\ \alpha &= \frac{1}{\sqrt{2}} \end{aligned}$$

Therefore, $\beta = \sqrt{1 - \alpha^2} = \frac{1}{\sqrt{2}}$ and hence $\alpha = \beta = \frac{1}{\sqrt{2}}$ which means that the rectangle is a square (noting that the square is a special case of rectangle in which the length and width are equal).

Note 1: according to the result of this Problem, the length of the longest perimeter of a rectangle inscribed in a circle of radius R is $4\sqrt{2}R$ (where this rectangle is a square).
Note 2: it is obvious that the optimal perimeter in this Problem is a maximum (not a minimum) because the perimeter can converge to $4R$ (i.e. 4 times the circle radius) when the width of the rectangle approaches zero.

6. What is the shape of the rectangle of optimal area inscribed inside a circle?
   Answer: If we follow the setting and reasoning of Problem 5 then the area σ of the rectangle is:

   $$\sigma = 4\alpha\beta = 4\alpha\sqrt{1-\alpha^2} = 4\sqrt{\alpha^2-\alpha^4}$$

   Accordingly, we are required to optimize $\sqrt{\alpha^2-\alpha^4}$ by taking its derivative with respect to α and setting it to zero to obtain the optimization condition, that is:

   $$\begin{aligned} \frac{d}{d\alpha}\left(\sqrt{\alpha^2-\alpha^4}\right) &= 0 \\ \frac{2\alpha-4\alpha^3}{2\sqrt{\alpha^2-\alpha^4}} &= 0 \\ \alpha-2\alpha^3 &= 0 \\ \alpha\left(1-2\alpha^2\right) &= 0 \\ 1-2\alpha^2 &= 0 \qquad\qquad (\alpha \neq 0) \\ \alpha &= \frac{1}{\sqrt{2}} \end{aligned}$$

   Therefore, $\beta = \sqrt{1-\alpha^2} = \frac{1}{\sqrt{2}}$ and hence $\alpha = \beta = \frac{1}{\sqrt{2}}$ which means that the rectangle is a square.
   Note 1: according to the result of this Problem, the maximum area of a rectangle inscribed in a circle of radius R is $2R^2$ (where this rectangle is a square).
   Note 2: it is obvious that the optimal area in this Problem is a maximum (not a minimum) because the area can converge to zero when the width of the rectangle approaches zero.
7. What is the shape of the rectangle of optimal perimeter inscribed inside an ellipse? Also, find the length of the optimal perimeter.
   Answer: Let use a Cartesian coordinate system centered on the center of the ellipse with the major axis of the ellipse being on the x axis and hence the sides of the rectangle being parallel to the coordinate axes (see Figure 38). Also, let the ellipse have semi-major axis a and semi-minor axis b. Due to the symmetry, the perimeter p of the rectangle is given by $p = 4(\alpha+\beta)$ where α and β are the coordinates of the vertex in the first quadrant. Hence, we are required to optimize $\alpha+\beta$ (because 4 is no more than a scale factor). Now, the vertex (α, β) is on the ellipse and hence it should satisfy the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, that is:

   $$\begin{aligned} \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2} &= 1 \\ \beta &= b\sqrt{1-\frac{\alpha^2}{a^2}} \end{aligned}$$

   Accordingly, we are required to optimize $\alpha + b\sqrt{1-\frac{\alpha^2}{a^2}}$ by taking its derivative with respect to α and setting it to zero to obtain the optimization condition, that is:

   $$\begin{aligned} \frac{d}{d\alpha}\left[\alpha + b\sqrt{1-\frac{\alpha^2}{a^2}}\right] &= 0 \\ 1 + \frac{b\left(-2\frac{\alpha}{a^2}\right)}{2\sqrt{1-\frac{\alpha^2}{a^2}}} &= 0 \end{aligned}$$

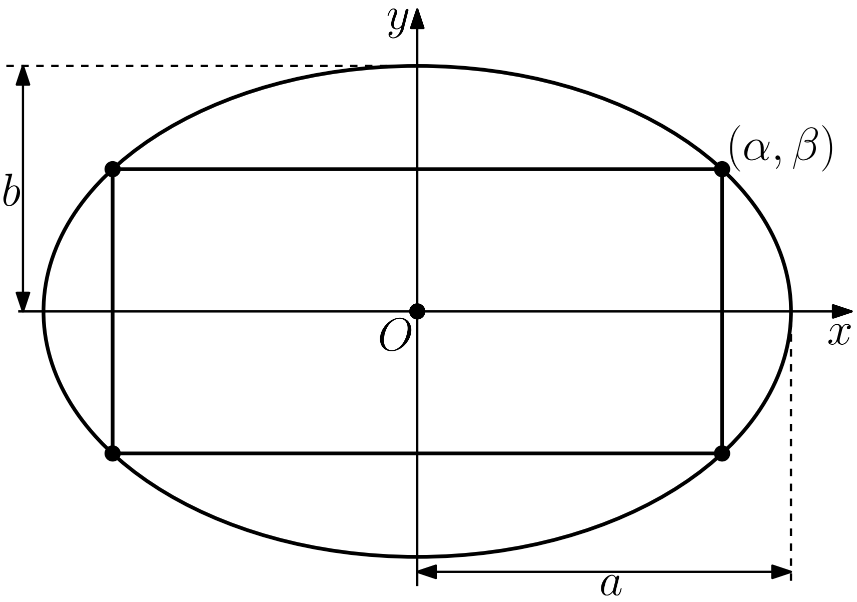

Figure 38: The setting of Problem 7 of § 3.3 where a rectangle is inscribed inside an ellipse (with semi-major axis a and semi-minor axis b) centered on the origin of coordinates. The major axis of the ellipse is on the x axis (and hence the sides of the rectangle are parallel to the coordinate axes), and α and β are the x and y coordinates of the rectangle vertex in the first quadrant.

$$\begin{aligned}
1-\frac{b\alpha}{a^2\sqrt{1-\frac{\alpha^2}{a^2}}} &= 0\\
1-\frac{b\alpha}{\sqrt{a^4-a^2\alpha^2}} &= 0\\
\frac{b^2\alpha^2}{a^4-a^2\alpha^2} &= 1\\
b^2\alpha^2+a^2\alpha^2 &= a^4\\
\alpha^2 &= \frac{a^4}{a^2+b^2}\\
\alpha &= \frac{a^2}{\sqrt{a^2+b^2}}
\end{aligned}$$

Therefore:

$$\beta = b\sqrt{1-\frac{\alpha^2}{a^2}} = b\sqrt{1-\frac{1}{a^2}\left(\frac{a^4}{a^2+b^2}\right)} = b\sqrt{1-\frac{a^2}{a^2+b^2}} = \frac{b^2}{\sqrt{a^2+b^2}}$$

Hence, the rectangle of optimal perimeter has length $L = 2\alpha = \frac{2a^2}{\sqrt{a^2+b^2}}$ and width $W = 2\beta = \frac{2b^2}{\sqrt{a^2+b^2}}$. So, its perimeter is:

$$p = 2L + 2W = \frac{4a^2}{\sqrt{a^2+b^2}} + \frac{4b^2}{\sqrt{a^2+b^2}} = \frac{4\left(a^2+b^2\right)}{\sqrt{a^2+b^2}} = 4\sqrt{a^2+b^2}$$

Note 1: it is obvious that the optimal perimeter in this Problem is a maximum (not a minimum) because the perimeter can converge to $4b$ when α approaches zero and it can converge to $4a$ when β approaches zero (noting that $4b < 4\sqrt{a^2+b^2}$ and $4a < 4\sqrt{a^2+b^2}$).
Note 2: if we set $a = b = R$ (i.e. the ellipse becomes a circle of radius R) then we obtain $\alpha = \beta = \frac{R}{\sqrt{2}}$,

i.e. we retrieve (as a special case) the result of the circle that we obtained in Problem 5 (noting that in Problem 5 we used *unit* circle).

8. What is the shape of the rectangle of optimal area inscribed inside an ellipse? Also, find the optimal area.
Answer: If we follow the setting and reasoning of Problem 7 then the area σ of the rectangle is:
$$\sigma = 4\alpha\beta = 4\alpha b\sqrt{1-\frac{\alpha^2}{a^2}} = \frac{4b}{a}\sqrt{a^2\alpha^2-\alpha^4}$$
Accordingly, we are required to optimize $\sqrt{a^2\alpha^2-\alpha^4}$ by taking its derivative with respect to α and setting it to zero to obtain the optimization condition, that is:
$$\begin{aligned} \frac{d}{d\alpha}\left(\sqrt{a^2\alpha^2-\alpha^4}\right) &= 0 \\ \frac{2a^2\alpha-4\alpha^3}{2\sqrt{a^2\alpha^2-\alpha^4}} &= 0 \\ a^2\alpha-2\alpha^3 &= 0 \\ \alpha\left(a^2-2\alpha^2\right) &= 0 \\ a^2-2\alpha^2 &= 0 \qquad\qquad (\alpha\neq 0) \\ \alpha &= \frac{a}{\sqrt{2}} \end{aligned}$$
Therefore:
$$\beta = b\sqrt{1-\frac{\alpha^2}{a^2}} = b\sqrt{1-\frac{1}{a^2}\left(\frac{a^2}{2}\right)} = b\sqrt{1-\frac{1}{2}} = \frac{b}{\sqrt{2}}$$
Hence, the rectangle of optimal area has length $L = 2\alpha = \sqrt{2}a$ and width $W = 2\beta = \sqrt{2}b$. So, its area is:
$$\sigma = LW = \sqrt{2}a\sqrt{2}b = 2ab$$
Note 1: it is obvious that the area in this Problem is a maximum (not a minimum) because the area can converge to zero when α or β approaches zero.
Note 2: if we set $a = b = R$ (i.e. the ellipse becomes a circle of radius R) then we obtain $\alpha = \beta = \frac{R}{\sqrt{2}}$, i.e. we retrieve (as a special case) the result of the circle that we obtained in Problem 6 (noting that in Problem 6 we used *unit* circle).
Note 3: comparing the result of this Problem to the result of Problem 7 we see that the optimal rectangles in these Problems are different. This is unlike the Corresponding Problems of rectangles inscribed in circles (see Problems 5 and 6) where the two "rectangles" are the same because they are actually squares due to the circular symmetry.

9. What is the shape of the triangle of optimal area whose two vertices are on the foci of an ellipse while the other vertex is on the perimeter of the ellipse?
Answer: According to the definition of ellipse,[69] the perimeter of this triangle is fixed and hence this Problem is an instance of Problem 5 of § 3.2 since the perimeter and one of the sides (i.e. the side that connects the two foci) of the triangle are fixed. Therefore, the triangle is an isosceles and its optimal area is a maximum (according to note 1 of Problem 5 of § 3.2).
Note 1: the optimal area in this Problem is $\sigma = bc = b\sqrt{a^2-b^2}$ where a is the semi-major axis of the ellipse, b is the semi-minor axis and c is the distance between the center and a focus.
Note 2: the result of the present Problem can be easily obtained from the fact that the area of triangle is half its base times its height plus the fact that in our case the base is fixed while the height takes its

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[69] As it should be known from elementary geometry, an ellipse is a plane curve with the property that the sum of the distances between any point on the curve and two fixed points (i.e. its foci) in the plane is constant (which is equal to $2a$ with a being the semi-major axis of the ellipse).

optimal (maximum) value when the two sides are equal (i.e. when the moving vertex of the triangle is at the co-vertex of the ellipse).[70]

10. What is the shape of the triangle of optimal perimeter whose two vertices are on the vertices of an ellipse[71] while the other vertex is on the perimeter of the ellipse?

    Answer: Referring to Figure 39, the side of the triangle on the major axis is fixed and hence we only need to optimize the sum S of the other two sides whose lengths are $\sqrt{h^2 + (a+X)^2}$ and $\sqrt{h^2 + (a-X)^2}$ (where $0 < h \le b$ and $0 \le X < a$), that is:

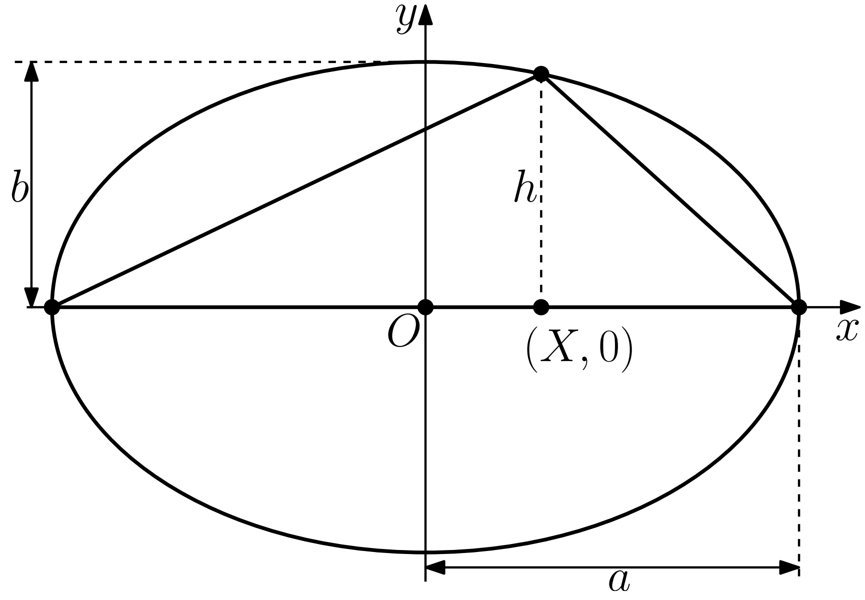

Figure 39: The setting of Problem 10 of § 3.3 where a triangle is inscribed inside an ellipse (with semi-major axis a and semi-minor axis b) centered on the origin of coordinates. The major axis of the ellipse (which is on the x axis) forms one side of the triangle while the opposite vertex of the triangle is on the ellipse perimeter.

$$S = \sqrt{h^2 + (a+X)^2} + \sqrt{h^2 + (a-X)^2}$$

Now, to optimize S we take its partial derivatives with respect to h and X and set them to zero to obtain the optimization conditions, that is:

$$\begin{aligned}
\frac{\partial S}{\partial h} &= 0 \\
\frac{h}{\sqrt{h^2 + (a+X)^2}} + \frac{h}{\sqrt{h^2 + (a-X)^2}} &= 0 \\
h\left[\frac{1}{\sqrt{h^2 + (a+X)^2}} + \frac{1}{\sqrt{h^2 + (a-X)^2}}\right] &= 0 \\
h &= 0 \qquad (86) \\
&\text{AND} \\
\frac{\partial S}{\partial X} &= 0
\end{aligned}$$

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[70] We mean by "co-vertex of the ellipse" the end point of the minor axis.

[71] We mean by "vertices of an ellipse" the end points of the major axis.

$$
\begin{aligned}
\frac{(a+X)}{\sqrt{h^2+(a+X)^2}}+\frac{-(a-X)}{\sqrt{h^2+(a-X)^2}} &= 0 \\
\frac{(a+X)^2}{h^2+(a+X)^2} &= \frac{(a-X)^2}{h^2+(a-X)^2} \\
(a+X)^2\left[h^2+(a-X)^2\right] &= (a-X)^2\left[h^2+(a+X)^2\right] \\
h^2(a+X)^2+(a+X)^2(a-X)^2 &= h^2(a-X)^2+(a+X)^2(a-X)^2 \\
h^2(a+X)^2 &= h^2(a-X)^2 \\
(a+X)^2 &= (a-X)^2 \qquad (h\neq 0) \\
a^2+2aX+X^2 &= a^2-2aX+X^2 \\
2aX &= -2aX \\
X &= -X \qquad (a\neq 0) \\
X &= 0 \qquad (87)
\end{aligned}
$$

As we see, Eq. 86 is unacceptable because $0<h\leq b$ (in fact $h=0$ corresponds to the lower limit of the perimeter). However, Eq. 87 is acceptable (since $0\leq X<a$) and hence the triangle is an isosceles (with $h=b$). Accordingly, the triangle in this Problem is an isosceles with an optimal perimeter of $2a+2\sqrt{a^2+b^2}$.

Note 1: if the fixed side in this Problem is on the minor axis then the result is similar, i.e. the triangle is an isosceles with an optimal perimeter of $2b+2\sqrt{a^2+b^2}$.

Note 2: it is obvious that the optimal perimeter in this Problem is a maximum (not a minimum) because the perimeter can converge to $4a$ (or $4b$) when the height h of the triangle approaches zero.

Note 3: the circle is a special case of ellipse and hence the result of this Problem also applies to circles, i.e. the triangle of optimal perimeter whose one side is a diameter of a circle while the opposite vertex is on the circumference of the circle is an isosceles of optimal (i.e. maximum) perimeter of $2R+2\sqrt{2}R$ (with R being the radius of the circle).

11. What is the shape of the triangle of optimal area whose two vertices are on the vertices of an ellipse while the other vertex is on the perimeter of the ellipse?

Answer: Referring to the setting of Problem 10 and Figure 39, the area σ of the triangle is half the base (which is $2a$) times the height (which is h) and hence $\sigma=ah$. Now, a is constant and hence the optimal of σ occurs when h is maximum (noting that $\sigma\neq 0$ and $h\neq 0$). Hence, from the condition $0<h\leq b$ we conclude that the optimal area is when $h=b$. Accordingly, the triangle in this Problem is an isosceles with an optimal (i.e. maximum) area of ab.

Note 1: if the fixed side in this Problem is on the minor axis then the result is similar, i.e. the triangle is an isosceles with an optimal (i.e. maximum) area of ab.

Note 2: the circle is a special case of ellipse and hence the result of this Problem also applies to circles, i.e. the triangle of optimal area whose one side is a diameter of a circle while the opposite vertex is on the circumference of the circle is an isosceles of optimal (i.e. maximum) area of R^2 (with R being the radius of the circle).

3.4 Surface of Revolution of Optimal Area

In this type of problems it is required to optimize the area of a surface of revolution[72] generated by rotating a planar curve Γ represented by the function $y=y(x)$ with fixed end points $A(x_A,y_A)$ and $B(x_B,y_B)$ around the x axis (see Figure 40).[73] So, the objective is to find the form of the planar curve that produces such a surface of revolution of optimal area. It should be noted that although the immediate objective here is to find the shape of a curve, the ultimate objective is to find the shape of a surface and hence the subject is classified as an optimal surface issue. This is due to the fact that we are dealing with a surface of revolution which is completely determined by its profile curve.

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[72] "Surface of revolution" is a surface generated by revolving a plane curve around a straight line in its plane.

[73] The representation of Γ by the function $y=y(x)$ is the common case. Other representations are also possible.

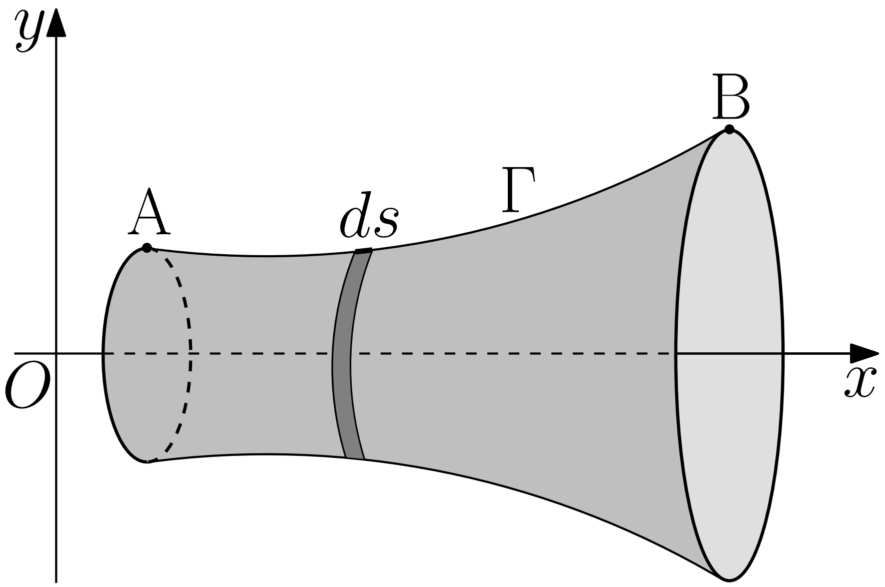

Figure 40: A simple sketch depicting the setting of Problem 1 of § 3.4 where a surface of revolution is generated by revolving a planar curve Γ around the x axis. The curve Γ is represented by a function $y(x)$ and it connects two fixed points (A and B) with ds representing the length of an infinitesimal arc generating an infinitesimal ring of radius y and width ds.

Problems

1. Find the shape of the surface of revolution of optimal area (as described above in the text).
   Answer: The area σ of a surface of revolution is the sum of the areas of the infinitesimal rings of radius y and width ds (see Figure 40). Noting that the area of each one of these rings (which are infinitesimal cylinders) is given by $2\pi y ds$ (i.e. perimeter $2\pi y$ times width ds), the area σ should be given by the following integral (which is the functional that we intend to minimize in this Problem):

   $$\sigma = \int_\Gamma 2\pi y ds = 2\pi \int_{x_A}^{x_B} y\sqrt{1+y'^2}\, dx \equiv I[y]$$

   Accordingly, $F(x, y, y') = y\sqrt{1+y'^2}$. As we see, F in this Problem is identical to F in the problem of catenary (see Problem 1 of § 2.3). Therefore, the shape of the curve Γ (which represents the profile of the required surface of revolution) should also be a catenary (i.e. hyperbolic cosine), that is:

   $$y = C\cosh\left(\frac{x-D}{C}\right) \tag{88}$$

   A surface generated by the revolution of a catenary (around its horizontal axis) is called catenoid, and hence the surface of revolution of optimal area is a catenoid.
   Note 1: the constants C and D in Eq. 88 can be determined from the two boundary conditions at the end points of the curve (see Problem 2).
   Note 2: the optimal solution of this Problem should be a minimum (not a maximum) because it is obvious that the surface area generated by the revolution of some curves can diverge and hence the optimal solution cannot be a maximum.[74] Also see Problem 4.
   Note 3: the existence and uniqueness of the solution of this Problem is not guaranteed. In more

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[74] Although this sort of arguments essentially rules out the possibility of global (or absolute) maximum, it should work in general even for local maximum (noting also our lax approach in pursuing issues like determining the nature of optimum and considering the upcoming Problem 4 as well although it is also based ultimately on a similar argument).

accurate terms, depending on the positions of the fixed end points (A and B) there may be one solution of this type (which is usually the required solution), or more than one solution (with only one possibly representing the required minimal surface) or there is no solution at all.[75]

2. Given that the profile curve of the surface of Problem 1 passes through the boundary points $(0, 1)$ and $(0.5, 2)$, find the equation of this profile curve and plot it.
Answer: We start by solving Eq. 88 for D, that is:

$$D = x - C \operatorname{arccosh}\left(\frac{y}{C}\right)$$

On substituting the coordinates of the two boundary points into this equation we get:

$$\begin{aligned} D &= 0 - C \operatorname{arccosh}\left(\frac{1}{C}\right) && (89) \\ D &= 0.5 - C \operatorname{arccosh}\left(\frac{2}{C}\right) && (90) \end{aligned}$$

On subtracting Eq. 89 from Eq. 90 we get:

$$0.5 - C \operatorname{arccosh}\left(\frac{2}{C}\right) + C \operatorname{arccosh}\left(\frac{1}{C}\right) = 0$$

On solving this equation for C (using a numerical solver) we get $C \simeq 0.6348247523$ and hence $D \simeq -0.6518922526$ (where we use Eq. 89 or Eq. 90 to find D). Accordingly, the equation of the profile curve is:

$$y \simeq 0.6348247523 \cosh\left(\frac{x + 0.6518922526}{0.6348247523}\right) \simeq 0.6348247523 \cosh\left(1.575237885x + 1.026885373\right)$$

The profile curve is plotted in Figure 41.[76]

3. Confirm the result of Problem 1 using cylindrical coordinates.
Answer: A surface of revolution (with a profile curve Γ) whose axis of symmetry is the z axis can be given in cylindrical coordinates as $\rho = \rho(z)$. Hence, its surface area is:

$$\sigma = \int_\Gamma 2\pi\rho ds = 2\pi \int_\Gamma \rho\sqrt{(d\rho)^2 + (dz)^2} = 2\pi \int_{z_1}^{z_2} \rho\sqrt{\rho'^2 + 1}\, dz \equiv I[\rho]$$

where $\rho' = d\rho/dz$. Accordingly, $F(z, \rho, \rho') = \rho\sqrt{\rho'^2 + 1}$ and hence the Euler-Lagrange equation is (see Eq. 3 noting the correspondence between z, ρ, ρ' here and x, y, y' there and the absence of z here):

$$\begin{aligned} \rho\sqrt{\rho'^2 + 1} - \rho' \frac{\partial}{\partial \rho'}\left[\rho\sqrt{\rho'^2 + 1}\right] &= C \\ \rho\sqrt{\rho'^2 + 1} - \frac{\rho\rho'^2}{\sqrt{\rho'^2 + 1}} &= C \\ \rho\left(\rho'^2 + 1\right) - \rho\rho'^2 &= C\sqrt{\rho'^2 + 1} \\ \rho &= C\sqrt{\rho'^2 + 1} \\ \rho' &= \sqrt{\left(\frac{\rho}{C}\right)^2 - 1} \end{aligned}$$

---

[75] In fact, there are many details here about the nature of the solution and if it exists or not. However, we do not go through these details since our primary objective in this book is the investigation of the techniques of variational calculus. The interested reader should refer to the literature (see for example Weinstock in the References).

[76] As in footnote [75], in this Problem and its alike we are only interested in investigating and demonstrating the techniques (rather than the nature of the solution and its physical significance as we have no primary interest in these details).

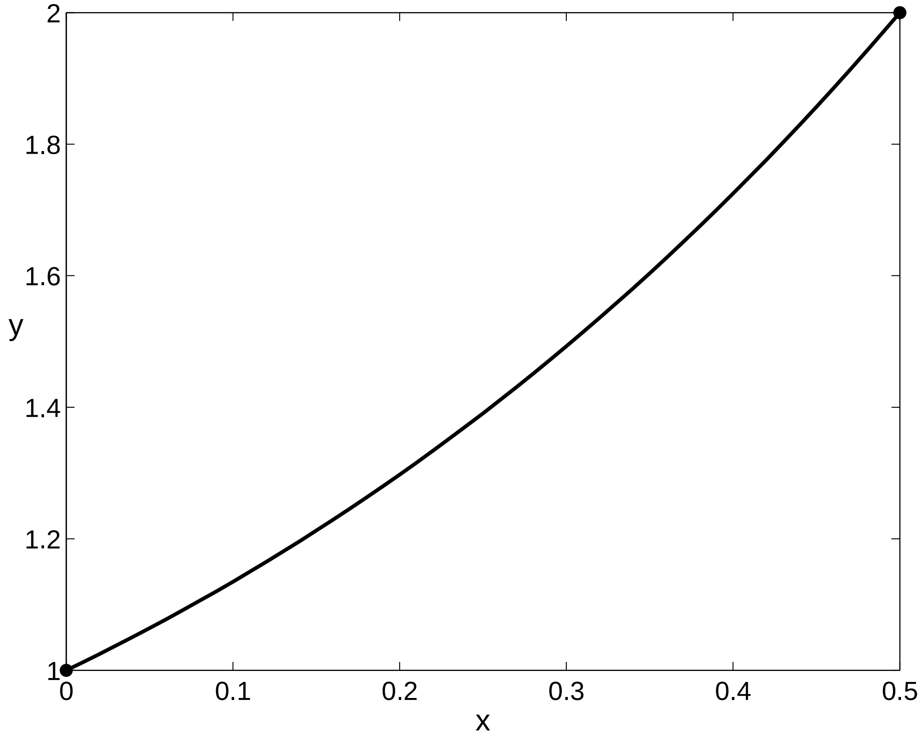

Figure 41: Plot of the profile curve of the minimum surface of revolution that passes through the boundary points $(0, 1)$ and $(0.5, 2)$. See Problem 2 of § 3.4.

$$\begin{aligned} dz &= \frac{d\rho}{\sqrt{\left(\frac{\rho}{C}\right)^2 - 1}} \\ z &= C \operatorname{arccosh}\left(\frac{\rho}{C}\right) + D \qquad (91) \\ \rho &= C \cosh\left(\frac{z - D}{C}\right) \end{aligned}$$

Now, if we note the above-indicated correspondence between z, ρ, ρ' and x, y, y' then the last equation is the same as Eq. 88, as required.

4. Verify the fact that catenoid is a minimal surface using the result of Problem 4 of § 1.6.
Answer: We use the settings of Problem 3 which satisfy the domain and boundary conditions of Problem 4 of § 1.6.[77] Hence, according to Eq. 91 (which is an equation of a catenoid in 3D as well as a catenary in 2D) we have:

$$\begin{aligned} z &= C \operatorname{arccosh}\left(\frac{\sqrt{x^2+y^2}}{C}\right) + D \qquad (\rho = \sqrt{x^2+y^2}) \\ z_x &= \frac{x}{\sqrt{x^2+y^2}\sqrt{\frac{x^2+y^2}{C^2} - 1}} \\ z_{xx} &= \frac{-x^4 + y^4 - C^2 y^2}{C^2 \left(x^2+y^2\right)^{3/2} \left(\frac{x^2+y^2}{C^2} - 1\right)^{3/2}} \end{aligned}$$

---

[77] In fact, there are some issues about the domain and boundary conditions that require discussion and clarification. However, we ignore these issues because they are not relevant or necessary for our objectives.

$$\begin{aligned}
z_y &= \frac{y}{\sqrt{x^2+y^2}\sqrt{\frac{x^2+y^2}{C^2}-1}} \\
z_{yy} &= \frac{-y^4+x^4-C^2x^2}{C^2\left(x^2+y^2\right)^{3/2}\left(\frac{x^2+y^2}{C^2}-1\right)^{3/2}} \\
z_{xy} &= \frac{-xy\left(2x^2+2y^2-C^2\right)}{C^2\left(x^2+y^2\right)^{3/2}\left(\frac{x^2+y^2}{C^2}-1\right)^{3/2}}
\end{aligned}$$

On substituting from these equations into Eq. 17 we get $0 = 0$, i.e. Eq. 17 is satisfied identically by the equation of the catenoid. So, the catenoid is a solution to Eq. 17 and hence its area is minimum (according to Problem 4 of § 1.6), i.e. it is a minimal surface, as required.

Chapter 4
Optimal Solids

In this chapter we present and solve problems about topics and applications of the mathematics of variation related to solids, i.e. we are looking in these problems to certain 3D objects that optimize something (such as volume).

4.1 3D Shapes of Optimal Sides Lengths

From the title of this section, it is fairly obvious that in this type of problems it is required to optimize the sum of the lengths of the sides of a given 3D shape[78] (such as rectangular parallelepiped) subject to certain condition(s) such as being of fixed surface area. We note that due to the simplicity of this type of problems we use only ordinary calculus (possibly in association with the Lagrange multipliers technique) in solving these problems and hence no variational formulation (based on the Euler-Lagrange equation) is required.

Problems

1. What is the shape of the rectangular parallelepiped of fixed surface area that optimizes the sum of the lengths of all its sides?
   Answer: This is a length optimization problem with an area constraint because we are required to optimize the sum of sides lengths subject to the surface area constraint. So, we can solve it by using the Lagrange multipliers technique (see § 1.8).
   If the three dimensions of the parallelepiped are x, y, z then its sum of sides lengths is $4(x+y+z)$ and its surface area is $2(xy+xz+yz)$. So, our Lagrange multipliers formulation is $f+\lambda g = 4(x+y+z)+\lambda 2(xy+xz+yz)$ and hence we should optimize $f+\lambda g$ by taking its partial derivatives with respect to x, y, z and setting them to zero to obtain the optimization conditions, that is:
$$\frac{\partial}{\partial x}\left[4(x+y+z)+\lambda 2(xy+xz+yz)\right] = 4+\lambda 2(y+z) = 0 \tag{92}$$
$$\frac{\partial}{\partial y}\left[4(x+y+z)+\lambda 2(xy+xz+yz)\right] = 4+\lambda 2(x+z) = 0 \tag{93}$$
$$\frac{\partial}{\partial z}\left[4(x+y+z)+\lambda 2(xy+xz+yz)\right] = 4+\lambda 2(x+y) = 0 \tag{94}$$
   On subtracting Eq. 93 from Eq. 92 we get $x = y$ while on subtracting Eq. 94 from Eq. 93 we get $y = z$. Hence, $x = y = z$ which means that our parallelepiped is a cube.
   Note: it should be obvious that the optimal sum of sides lengths in this Problem is a minimum (not a maximum) because this sum will diverge if two of the dimensions (say y and z) of the parallelepiped approach zero (noting that the surface area is fixed).
2. What is the shape of the rectangular parallelepiped of fixed volume that optimizes the sum of the lengths of all its sides?
   Answer: This is a length optimization problem with a volume constraint. So, we can solve it by the Lagrange multipliers technique (see § 1.8) as we did in Problem 1 (see the upcoming note 1). However, for diversity and comparison we solve it using a different method, i.e. by embedding the volume constraint within the expression of the optimized quantity (which is the sum of sides lengths). According to the volume constraint we have $V = xyz$ (where x, y, z are the three dimensions of the

---

[78] More technically and specifically, the 3D shapes of interest in this section have straight sides and hence they are polyhedrons. So, in these problems we are optimizing the sum of the lengths of the sides of polyhedrons.

parallelepiped and V is the volume which is a constant) and hence $z = \frac{V}{xy}$. So, the sum of sides lengths is $S = 4(x+y+z) = 4\left(x + y + \frac{V}{xy}\right)$. Now, we should optimize S by taking its partial derivatives with respect to x and y and setting them to zero to obtain the optimization conditions, that is:

$$\begin{aligned}\frac{1}{4}\frac{\partial S}{\partial x} &= 1 - \frac{V}{x^2 y} = 0\\ \frac{1}{4}\frac{\partial S}{\partial y} &= 1 - \frac{V}{xy^2} = 0\end{aligned}$$

On subtracting the first equation from the second we get:

$$\begin{aligned}\frac{V}{x^2 y} - \frac{V}{xy^2} &= 0\\ x^2 y &= xy^2\\ x &= y \qquad \text{(dividing by } xy \neq 0)\end{aligned}$$

Now, if we repeat the above process but this time with $y = \frac{V}{xz}$ (instead of $z = \frac{V}{xy}$) then we get $x = z$. Hence, $x = y = z$ which means that our parallelepiped is a cube.

Note 1: if we use the Lagrange multipliers technique then $f + \lambda g = 4(x + y + z) + \lambda xyz$ and hence:

$$\begin{aligned}\frac{\partial}{\partial x}(f + \lambda g) &= 4 + \lambda yz = 0\\ \frac{\partial}{\partial y}(f + \lambda g) &= 4 + \lambda xz = 0\\ \frac{\partial}{\partial z}(f + \lambda g) &= 4 + \lambda xy = 0\end{aligned}$$

which by comparison lead to $x = y = z$.

Note 2: it should be obvious that the optimal sum of sides lengths in this Problem is a minimum (not a maximum) because this sum will diverge if two of the dimensions (say y and z) of the parallelepiped approach zero (noting that the volume is fixed).[79]

3. What is the shape of the regular pyramid of square base and fixed volume that optimizes the sum of the lengths of all its sides?[80]

   Answer: Let a be the length of the 4 sides of the square base, A the length of the 4 slant sides, V the fixed volume of the pyramid, H its height, and h the height of the slant faces (see Figure 42). Accordingly, we have:

$$V = \frac{1}{3}a^2 H \qquad \text{and hence} \qquad H = \frac{3V}{a^2}$$

   Now, the sum S of the lengths of all sides is given by:

$$\begin{aligned}S &= 4a + 4A\\ &= 4a + 4\sqrt{\frac{a^2}{2} + H^2}\\ &= 4a + 4\sqrt{\frac{a^2}{2} + \left(\frac{3V}{a^2}\right)^2}\\ &= 4\left(a + \sqrt{\frac{a^2}{2} + \frac{9V^2}{a^4}}\right)\end{aligned}$$

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[79] In fact, it will diverge even if one dimension approaches zero.

[80] "Regular" means that all the 4 slant sides of the pyramid are identical.

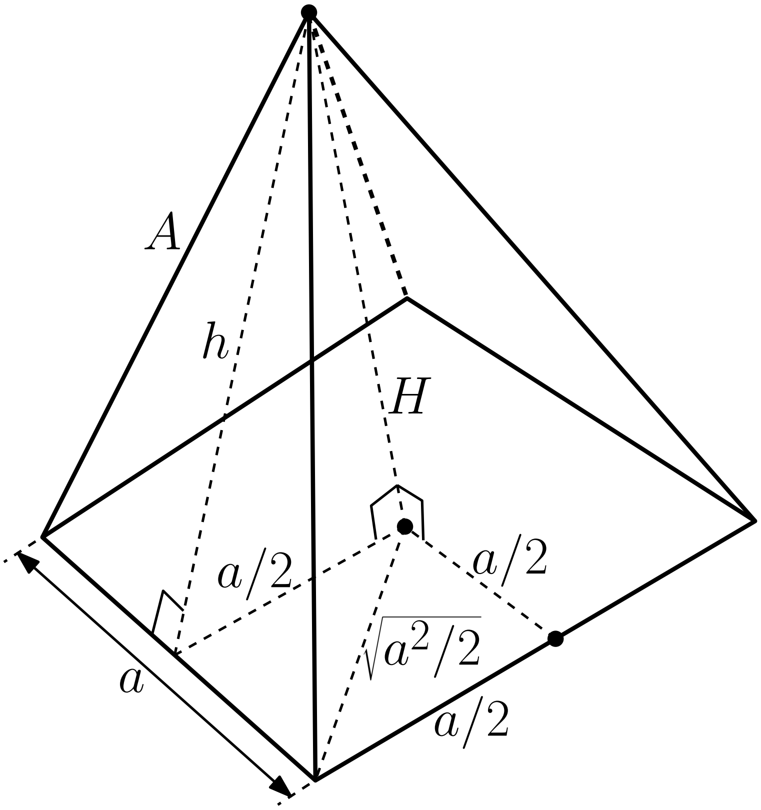

Figure 42: A schematic illustration of the setting of Problem 3 of § 4.1 showing a regular pyramid of square base and fixed volume. The symbols are explained in the text.

To optimize S we take its derivative with respect to a and set it to zero to obtain the optimization condition, that is:

$$\begin{aligned}
\frac{1}{4}\frac{dS}{da} &= 0 \\
1+\frac{a-4\frac{9V^2}{a^5}}{2\sqrt{\frac{a^2}{2}+\frac{9V^2}{a^4}}} &= 0 \\
1+\frac{a^6-36V^2}{2a^5\sqrt{\frac{a^2}{2}+\frac{9V^2}{a^4}}} &= 0 \\
\left(a^6-36V^2\right)^2 &= 4a^{10}\left(\frac{a^2}{2}+\frac{9V^2}{a^4}\right) \\
a^{12}-72V^2a^6+36^2V^4 &= 2a^{12}+36V^2a^6 \\
a^{12}+108V^2a^6-36^2V^4 &= 0
\end{aligned}$$

On solving the last equation for the variable a^6 (using the quadratic formula), we get:

$$a^6=\frac{-108V^2\pm\sqrt{108^2V^4+4\times 36^2V^4}}{2}=\left(-54\pm 18\sqrt{13}\right)V^2$$

The only physically[81] sensible root is the positive and hence $a^6=\left(-54+18\sqrt{13}\right)V^2$ and thus $a=$

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[81] In fact, even mathematically considering our restriction to real quantities.

$\left(-54+18\sqrt{13}\right)^{1/6}V^{1/3}$. So, our pyramid has 4 base sides each of length $a=\left(-54+18\sqrt{13}\right)^{1/6}V^{1/3}$ and 4 slant sides each of length $A=\sqrt{\frac{a^2}{2}+\frac{9V^2}{a^4}}$ (with a being given by the previous expression).
Note: it should be obvious that the optimal sum of sides lengths in this Problem is a minimum (not a maximum) because this sum will diverge if the height H of the pyramid approaches zero or if the base shrinks (noting that the volume is fixed).

4.2 3D Shapes of Optimal Surface Area

From the title of this section, it is obvious that in this type of problems it is required to optimize the surface area of a given 3D shape (such as pyramid or cone) subject to certain condition(s) such as being of fixed volume. As in the Problems of § 4.1, we use only ordinary calculus in solving these problems without need for employing the variational formulation of the Euler-Lagrange equation.

Problems

1. What is the shape of the regular pyramid of square base and fixed volume that optimizes the surface area?
   Answer: If we follow the setting of Problem 3 of § 4.1 (including Figure 42) then the surface area σ of the pyramid is the sum of the base area which is a^2 plus 4 triangular slant faces each of area $\frac{1}{2}ah$. Hence, we have:

$$\begin{aligned}\sigma &= a^2+4\left(\frac{1}{2}a\sqrt{\frac{a^2}{4}+H^2}\right)\\ &= a^2+2a\sqrt{\frac{a^2}{4}+\left(\frac{3V}{a^2}\right)^2}\\ &= a^2+2\sqrt{\frac{a^4}{4}+\frac{9V^2}{a^2}}\end{aligned}$$

   To optimize σ we take its derivative with respect to a and set it to zero to obtain the optimization condition, that is:

$$\begin{aligned}\frac{d\sigma}{da} &= 0\\ 2a+2\frac{a^3-2\frac{9V^2}{a^3}}{2\sqrt{\frac{a^4}{4}+\frac{9V^2}{a^2}}} &= 0\\ 2a+\frac{a^6-18V^2}{a^3\sqrt{\frac{a^4}{4}+\frac{9V^2}{a^2}}} &= 0\\ \frac{\left(a^6-18V^2\right)^2}{a^6\left(\frac{a^4}{4}+\frac{9V^2}{a^2}\right)} &= 4a^2\\ a^{12}-36V^2a^6+18^2V^4 &= a^{12}+36V^2a^6\\ 72V^2a^6 &= 18^2V^4\\ a^6 &= \frac{9V^2}{2}\\ a &= \left(\frac{9}{2}\right)^{1/6}V^{1/3} \qquad (95)\end{aligned}$$

   So, our pyramid has a square base of side length a and height $H=\frac{3V}{a^2}$ with an optimal surface area $\sigma=a^2+2\sqrt{\frac{a^4}{4}+\frac{9V^2}{a^2}}$ (where a is given by Eq. 95).

Note: it should be obvious that the optimal surface area in this Problem is a minimum (not a maximum) because the surface area will diverge if the height H of the pyramid approaches zero (noting that the volume is fixed).

2. What is the shape of the right circular cone of fixed volume and optimal surface area?
Answer: This is an area optimization problem with a volume constraint, so we solve it by the Lagrange multipliers technique (see § 1.8). If the radius of the cone base is R and its height is H then its surface area is $\sigma = \pi R^2 + \pi R\sqrt{H^2+R^2}$ while its volume is $V = \frac{\pi}{3}R^2H$. So, what we should optimize (according to the Lagrange multipliers technique) is:

$$f + \lambda g \equiv \sigma + \lambda V = \pi R^2 + \pi R\sqrt{H^2+R^2} + \lambda\frac{\pi}{3}R^2H$$

Accordingly, we take the partial derivatives of $\sigma + \lambda V$ with respect to R and H (which are the variables in this optimization since they control the area and volume) and set them to zero to obtain the optimization conditions, that is:

$$\begin{aligned}\frac{1}{\pi}\frac{\partial}{\partial R}(\sigma+\lambda V) &= 2R + \sqrt{H^2+R^2} + \frac{R^2}{\sqrt{H^2+R^2}} + \frac{2\lambda}{3}RH = 0\\ \frac{1}{\pi}\frac{\partial}{\partial H}(\sigma+\lambda V) &= \frac{RH}{\sqrt{H^2+R^2}} + \frac{\lambda}{3}R^2 = 0\end{aligned}$$

Now, if we multiply the second equation by $\frac{2H}{R}$ and subtract the result from the first equation we get:

$$\begin{aligned}2R + \sqrt{H^2+R^2} + \frac{R^2}{\sqrt{H^2+R^2}} - \frac{2H^2}{\sqrt{H^2+R^2}} &= 0\\ 2R + \frac{H^2+R^2}{\sqrt{H^2+R^2}} + \frac{R^2}{\sqrt{H^2+R^2}} - \frac{2H^2}{\sqrt{H^2+R^2}} &= 0\\ 2R + \frac{2R^2-H^2}{\sqrt{H^2+R^2}} &= 0\\ \frac{\left(2R^2-H^2\right)^2}{H^2+R^2} &= 4R^2\\ 4R^4 - 4R^2H^2 + H^4 &= 4R^2H^2 + 4R^4\\ H^4 - 8R^2H^2 &= 0\\ H^2 - 8R^2 &= 0 \qquad (H \neq 0)\\ H &= \sqrt{8}R\end{aligned}$$

So, the optimal cone has a base radius $R = \left(\frac{3V}{\pi\sqrt{8}}\right)^{1/3}$ and height $H = \sqrt{8}R$ with an optimal surface area $\sigma = 4\pi R^2$.
Note: the optimal surface area in this Problem is a minimum because σ can diverge when H approaches zero (noting that the volume is fixed).

3. What is the shape of the rectangular parallelepiped of fixed volume and optimal surface area?
Answer: This is an area optimization problem with a volume constraint and hence we can solve it using the Lagrange multipliers technique (see § 1.8). Now, if the three dimensions of the parallelepiped are x, y, z then its surface area is $\sigma = 2(xy + xz + yz)$ and its volume is $V = xyz$. So, our Lagrange multipliers formulation is $f + \lambda g \equiv \sigma + \lambda V = 2(xy + xz + yz) + \lambda xyz$ and hence we should optimize $\sigma + \lambda V$ by taking its partial derivatives with respect to x, y, z and setting these partial derivatives to zero to obtain the optimization conditions, that is:

$$\frac{\partial}{\partial x}(\sigma+\lambda V) = 2(y+z) + \lambda yz = 0 \tag{96}$$

$$\frac{\partial}{\partial y}(\sigma+\lambda V) = 2(x+z) + \lambda xz = 0 \tag{97}$$

$$\frac{\partial}{\partial z}(\sigma + \lambda V) = 2(x+y) + \lambda xy = 0 \tag{98}$$

Now, if we multiply Eq. 96 with x and Eq. 97 with y and subtract the second from the first we get:

$$\begin{aligned} x\left[2(y+z) + \lambda yz\right] - y\left[2(x+z) + \lambda xz\right] &= 0 \\ 2x(y+z) - 2y(x+z) &= 0 \\ 2xz - 2yz &= 0 \\ x - y &= 0 \qquad\qquad (z \neq 0) \\ x &= y \end{aligned}$$

Similarly, if we multiply Eq. 97 with y and Eq. 98 with z and subtract the second from the first we get $y = z$.
On combining the results $x = y$ and $y = z$ we get $x = y = z$ which means that our parallelepiped is a cube.
Note: the optimal surface area in this Problem is a minimum because σ can diverge when one dimension (say z) approaches zero (noting that the volume is fixed).

4. What is the shape of the rectangular parallelepiped of fixed sum of sides lengths and optimal surface area?
Answer: This Problem can be solved by a similar method to that used in Problem 1 of § 4.1. However, it is easier to use the proof by contradiction to show that the shape is a cube because if the cube with fixed sum of sides lengths (say S_0) is not optimal (i.e. maximum; see the upcoming note) in surface area then there should be a non-cube parallelepiped with sum S_0 and with maximum surface area (say σ_0) and hence a cube of surface area σ_0 will have a larger sum than S_0 which contradicts the result of Problem 1 of § 4.1 because the cube of surface area σ_0 has minimum sum of sides lengths.[82]
Note: the optimal surface area in this Problem is a maximum because the surface area will converge to zero if two dimensions (say y and z) of the parallelepiped approach zero (noting that the sum of sides lengths is fixed).
5. What is the shape of the right circular cylinder of fixed volume and optimal surface area?
Answer: The surface area of right circular cylinder is $\sigma = 2\pi R^2 + 2\pi RH$ and its volume is $V = \pi R^2 H$ where R and H are its radius and height. We are asked to optimize σ subject to a constraint on V and hence according to the Lagrange multipliers technique we have $f + \lambda g = \sigma + \lambda V = 2\pi R^2 + 2\pi RH + \lambda \pi R^2 H$. So, we take the partial derivatives of $\sigma + \lambda V$ with respect to R and H and set them to zero to obtain the optimization conditions, that is:

$$\begin{aligned} \frac{1}{\pi}\frac{\partial}{\partial R}(\sigma + \lambda V) &= 4R + 2H + 2\lambda RH = 0 \\ \frac{1}{\pi}\frac{\partial}{\partial H}(\sigma + \lambda V) &= 2R + \lambda R^2 = 0 \end{aligned}$$

Now, if we multiply the second equation with $\frac{2H}{R}$ and subtract it from the first equation we get:

$$\begin{aligned} 4R + 2H - 4H &= 0 \\ H &= 2R \end{aligned}$$

So, the optimal cylinder has a height equal to its diameter with an optimal surface area $\sigma = 2\pi R^2 + 2\pi R\,(2R) = 6\pi R^2$ where $R = \left(\frac{V}{2\pi}\right)^{1/3}$.
Note: the optimal surface area in this Problem is a minimum because σ will diverge when H approaches zero (noting that the volume is fixed).

---

[82] For cube, $S = 12x$ and $\sigma = 6x^2$ (where S is the sum of sides lengths, σ is its surface area, and x is the length of its sides) which can be combined to obtain $S = 12\sqrt{\sigma/6}$ and hence the sum of sides lengths increases/decreases as the surface area increases/decreases (and vice versa).

4.3 3D Shapes of Optimal Volume

From the title of this section, it is clear that in this type of problems it is required to optimize the volume of a given 3D shape (such as right circular cylinder) subject to certain condition(s) such as being of fixed surface area. Again, we use only ordinary calculus in solving these problems with no need for the Euler-Lagrange equation.

Problems

1. What is the shape of the rectangular parallelepiped of fixed surface area and optimal volume?
   Answer: This is a volume optimization problem with an area constraint and hence it can be solved by the method of Problem 3 of § 4.2 where in this case we have $f+\lambda g \equiv V+\lambda\sigma = xyz+\lambda 2(xy+xz+yz)$ and hence by differentiation and elimination (as we did in Problem 3 of § 4.2) we again obtain $x=y=z$ which means that our parallelepiped is a cube.
   Note 1: the optimal volume in this Problem is a maximum because V will converge to zero when one dimension (say z) approaches zero (noting that the surface area is fixed).
   Note 2: the result of this Problem can be obtained as a corollary of the result of Problem 3 of § 4.2 (using the proof by contradiction method) because if the parallelepiped of the given surface area (say σ_0) and maximum volume (say V_0) was not cube then the cube of the surface area σ_0 should have smaller volume than V_0 and hence the cube of volume V_0 should have larger surface area than σ_0 and this contradicts the result of Problem 3 of § 4.2 because the cube of volume V_0 should have minimum surface area (in comparison to any other rectangular parallelepiped of the same volume).[83]
   Similarly, the result of Problem 3 of § 4.2 can be obtained as a corollary of the result of the present Problem (using the proof by contradiction method) because if the parallelepiped of the given volume (say V_0) and minimum surface area (say σ_0) was not cube then the cube of the volume V_0 should have larger surface area than σ_0 and hence the cube of surface area σ_0 should have smaller volume than V_0 and this contradicts the result of the present Problem because the cube of surface area σ_0 should have maximum volume (in comparison to any other parallelepiped of the same surface area).[84]
2. Find the volume of the rectangular parallelepiped of maximum volume with a surface area of 24 (area units).
   Answer: From the result of Problem 1, the parallelepiped should be a cube and hence from the formula of the surface area σ of cube we have $\sigma = 6x^2 = 24$ (where x is the length of its sides) which leads to $x = 2$. Now, from the formula of the volume V of cube we have $V = x^3 = 8$ (volume units).
3. What is the shape of the rectangular parallelepiped of fixed sum of sides lengths and optimal volume?
   Answer: This Problem can be solved by a similar method to that used in Problem 2 of § 4.1. However, it is easier to use the proof by contradiction to show that the shape is a cube because if the cube with fixed sum of sides lengths (say S_0) is not optimal (i.e. maximum; see the upcoming note) in volume then there should be a non-cube parallelepiped with sum S_0 and with maximum volume (say V_0) and hence a cube of volume V_0 will have a larger sum than S_0 which contradicts the result of Problem 2 of § 4.1 because the cube of volume V_0 has minimum sum of sides lengths.[85]
   Note: the optimal volume in this Problem is a maximum because for a parallelepiped with fixed sum of sides lengths if two dimensions (say y and z) approach zero the volume converges to zero.[86]
4. What is the shape of the right circular cone of fixed surface area and optimal volume?

---

[83] For cube, $\sigma = 6x^2$ and $V = x^3$ (where σ is its surface area, V is its volume, and x is the length of its sides) which can be combined to obtain $\sigma = 6V^{2/3}$ and hence the surface area increases/decreases as the volume increases/decreases (and vice versa).

[84] For cube, $V = x^3$ and $\sigma = 6x^2$ (where V is its volume, σ is its surface area, and x is the length of its sides) which can be combined to obtain $V = (\sigma/6)^{3/2}$ and hence the volume increases/decreases as the surface area increases/decreases (and vice versa).

[85] For cube, $S = 12x$ and $V = x^3$ (where S is the sum of sides lengths, V is its volume, and x is the length of its sides) which can be combined to obtain $S = 12V^{1/3}$ and hence the sum of sides lengths increases/decreases as the volume increases/decreases (and vice versa).

[86] In fact, it will converge to zero even if only one dimension approaches zero.

Answer: If we follow in our footsteps in Problem 2 of § 4.2 then we have:

$$f + \lambda g \equiv V + \lambda\sigma = \frac{\pi}{3}R^2H + \lambda\pi R^2 + \lambda\pi R\sqrt{H^2+R^2}$$

Hence:

$$\begin{aligned}
\frac{1}{\pi}\frac{\partial}{\partial R}(V+\lambda\sigma) &= \frac{2}{3}RH + 2\lambda R + \lambda\sqrt{H^2+R^2} + \frac{\lambda R^2}{\sqrt{H^2+R^2}} = 0 \\
\frac{1}{\pi}\frac{\partial}{\partial H}(V+\lambda\sigma) &= \frac{1}{3}R^2 + \frac{\lambda RH}{\sqrt{H^2+R^2}} = 0
\end{aligned}$$

Now, if we multiply the second equation by $\frac{2H}{R}$ and subtract the result from the first equation we get:

$$\begin{aligned}
2\lambda R + \lambda\sqrt{H^2+R^2} + \frac{\lambda R^2}{\sqrt{H^2+R^2}} - \frac{2\lambda H^2}{\sqrt{H^2+R^2}} &= 0 \\
2R + \sqrt{H^2+R^2} + \frac{R^2}{\sqrt{H^2+R^2}} - \frac{2H^2}{\sqrt{H^2+R^2}} &= 0 \qquad (\lambda \neq 0) \\
2R + \frac{H^2+R^2}{\sqrt{H^2+R^2}} + \frac{R^2}{\sqrt{H^2+R^2}} - \frac{2H^2}{\sqrt{H^2+R^2}} &= 0 \\
2R + \frac{2R^2-H^2}{\sqrt{H^2+R^2}} &= 0 \\
H &= \sqrt{8}R
\end{aligned}$$

which is the same as the result of Problem 2 of § 4.2 (so the optimal cone has a base radius $R = \sqrt{\frac{\sigma}{4\pi}}$ and height $H = \sqrt{8}R$ with an optimal volume $V = \frac{\pi\sqrt{8}}{3}R^3$). This should be the case to avoid contradiction because if a cone optimizes (i.e. minimizes) the surface area for a fixed volume then it should optimize (i.e. maximize) the volume for a fixed surface area. In fact, we could have established the result of this Problem using the proof by contradiction method in conjunction with the result of Problem 2 of § 4.2, but we followed this method for practice and confirmation.

Note: the optimal volume in this Problem is a maximum because V will converge to zero when H converges to zero (noting that the surface area is fixed).

5. What is the shape of the right circular cylinder of fixed surface area and optimal volume?

Answer: We can use the proof by contradiction in conjunction with the result of Problem 5 of § 4.2 to show that the optimal cylinder has a height equal to its diameter. However, for more practice we use the Lagrange multipliers technique again with $f + \lambda g = V + \lambda\sigma = \pi R^2H + 2\pi\lambda R^2 + 2\pi\lambda RH$. So, we take the partial derivatives of $V + \lambda\sigma$ with respect to R and H and set them to zero to obtain the optimization conditions, that is:

$$\begin{aligned}
\frac{1}{\pi}\frac{\partial}{\partial R}(V+\lambda\sigma) &= 2RH + 4\lambda R + 2\lambda H = 0 \\
\frac{1}{\pi}\frac{\partial}{\partial H}(V+\lambda\sigma) &= R^2 + 2\lambda R = 0
\end{aligned}$$

Now, if we multiply the second equation with $\frac{2H}{R}$ and subtract the result from the first equation we get:

$$\begin{aligned}
4\lambda R + 2\lambda H - 4\lambda H &= 0 \\
H &= 2R
\end{aligned}$$

So, the optimal cylinder has a height equal to its diameter with an optimal volume $V = \pi R^2(2R) = 2\pi R^3$ where $R = \sqrt{\frac{\sigma}{6\pi}}$.

Note: the optimal volume in this Problem is a maximum because V can approach zero when H approaches zero (noting that the surface area is fixed).

6. What is the shape of the ellipsoid of fixed sum of semi-axes lengths and optimal volume?
   Answer: The sum of the lengths of semi-axes a, b, c is $S = a+b+c$ while the volume is $V = \frac{4}{3}\pi abc$. It is required to optimize V subject to a constraint on S and hence we can use the Lagrange multipliers technique with $f + \lambda g \equiv V + \lambda S = \frac{4}{3}\pi abc + \lambda (a + b + c)$. On taking the partial derivatives of $V + \lambda S$ with respect to a, b, c and setting them to zero we can obtain the optimization conditions, that is:

$$\begin{aligned}\frac{\partial}{\partial a}(V+\lambda S) &= \frac{4}{3}\pi bc + \lambda = 0\\ \frac{\partial}{\partial b}(V+\lambda S) &= \frac{4}{3}\pi ac + \lambda = 0\\ \frac{\partial}{\partial c}(V+\lambda S) &= \frac{4}{3}\pi ab + \lambda = 0\end{aligned}$$

   By subtracting the second equation from the first equation we get $a = b$ while by subtracting the third equation from the first equation we get $a = c$. Hence, $a = b = c$ which means that the optimal ellipsoid is a sphere.
   Note: the optimal volume in this Problem is a maximum because V can approach zero when one (or two) of the semi-axes approach zero (noting that the sum of semi-axes lengths is fixed).
7. What is the shape of the right circular cone of fixed slant height (or generator) and optimal volume?
   Answer: Referring to Figure 43, if the fixed slant height is H while the height and base radius are h and r then $r = \sqrt{H^2 - h^2}$ and hence the volume of the cone is:

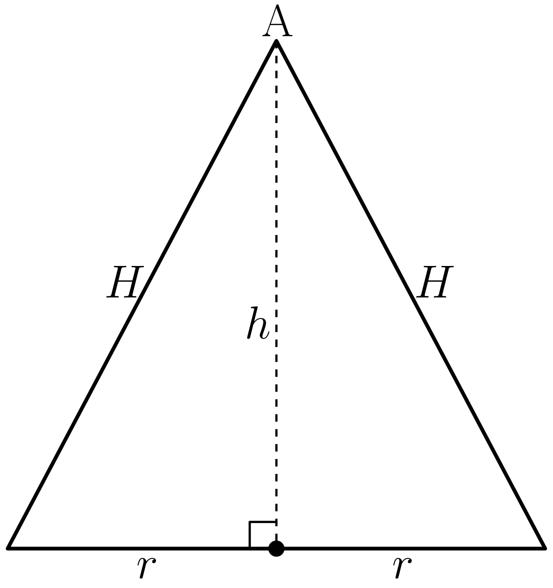

Figure 43: A cross section of a right circular cone of base radius r and height h with fixed slant height H. The cross section is through the apex A of the cone and the diameter of its base. See Problem 7 of § 4.3.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(H^2 - h^2\right)h = \frac{1}{3}\pi\left(H^2 h - h^3\right)$$

To optimize V we take its derivative with respect to h (which is the variable since H is fixed) and set the derivative to zero to obtain the optimization condition, that is:

$$\begin{aligned}\frac{3}{\pi}\frac{dV}{dh} &= 0\\ H^2 - 3h^2 &= 0\end{aligned}$$

$$h = \frac{H}{\sqrt{3}} \qquad (h > 0)$$

Hence, the optimal cone has $h = \frac{H}{\sqrt{3}}$ and $r = \sqrt{H^2 - \frac{H^2}{3}} = \sqrt{\frac{2}{3}}H$ with an optimal volume $V = \frac{1}{3}\pi\left(\frac{2}{3}H^2\right)\left(\frac{H}{\sqrt{3}}\right) = \frac{2\pi}{9\sqrt{3}}H^3$.

Note: the optimal volume in this Problem is a maximum because V can converge to zero when r approaches zero or when h approaches zero (noting that the slant height is fixed).

4.4 3D Shapes Inside Other 3D Shapes

In this type of problems it is required to optimize a certain property (such as volume) of a given 3D shape (such as rectangular parallelepiped) that is confined inside another 3D shape (such as ellipsoid). Again, no variational formulation (based on the Euler-Lagrange equation) is required because these problems are sufficiently simple to be solved by ordinary calculus.

Problems

1. What is the shape of the rectangular parallelepiped of optimal volume that is inscribed inside an ellipsoid of a fixed shape (noting that all the sides of the parallelepiped are parallel to the axes of the ellipsoid)?

   Answer: Using a rectangular Cartesian coordinate system, an ellipsoid in standard form (i.e. it is centered on the origin of coordinates and its three axes are aligned along the axes of the coordinate system) is given by the following equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{99}$$

   where a, b, c are the lengths of the semi-axes (which are positive constants). Now, by symmetry[87] the volume of the parallelepiped in each octant is equal to the volume in any other octant and hence if we consider the part of the parallelepiped that is in the first octant (i.e. the octant with $x, y, z > 0$) then the volume of this part is xyz (with x, y, z being the coordinates of the parallelepiped vertex in that octant) and hence the volume of the parallelepiped is $8xyz$ (since in 3D we have 8 octants).[88] So, our objective is to optimize $f = xyz$ (which is equivalent to optimizing $8xyz$). However, because the parallelepiped is inscribed inside the ellipsoid then its vertices should be on the ellipsoid and hence the x, y, z of the vertex in the first octant should satisfy the equation of the ellipsoid (i.e. Eq. 99). So in brief, we have to optimize xyz subject to the constraint of Eq. 99 and hence we can use the Lagrange multipliers technique by optimizing $h = f + \lambda g = xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$. Accordingly, we take the partial derivatives of h with respect to x, y, z (which are the variables of optimization since they control the volume of the parallelepiped) and set them to zero to obtain the optimization conditions, that is:

$$\frac{\partial h}{\partial x} = yz + \frac{2\lambda x}{a^2} = 0 \tag{100}$$

$$\frac{\partial h}{\partial y} = xz + \frac{2\lambda y}{b^2} = 0 \tag{101}$$

$$\frac{\partial h}{\partial z} = xy + \frac{2\lambda z}{c^2} = 0 \tag{102}$$

---

[87] The symmetry of the ellipsoid (as a result of being in standard form) should imply the symmetry of the parallelepiped, i.e. it is in standard form with its center being on the origin of coordinates and its sides being parallel to the axes of the coordinate system.

[88] The reader should note that we are actually using x, y, z in two different meanings, i.e. as general coordinates (as in Eq. 99) and as coordinates of the parallelepiped vertex in the first octant. The reason is to simplify the notation and avoid unnecessary complications; otherwise we can (for instance) use X, Y, Z for the coordinates of the vertex to distinguish between the two meanings.

On multiplying Eq. 100 with x and Eq. 101 with y and subtracting the results we get:

$$\begin{aligned} \frac{2\lambda x^2}{a^2} - \frac{2\lambda y^2}{b^2} &= 0 \\ \frac{x^2}{a^2} &= \frac{y^2}{b^2} \qquad (103) \end{aligned}$$

Also, on multiplying Eq. 100 with x and Eq. 102 with z and subtracting the results we get:

$$\begin{aligned} \frac{2\lambda x^2}{a^2} - \frac{2\lambda z^2}{c^2} &= 0 \\ \frac{x^2}{a^2} &= \frac{z^2}{c^2} \qquad (104) \end{aligned}$$

On comparing Eqs. 103 and 104 we get:

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

On substituting from the last equation into Eq. 99 (once for $\frac{y^2}{b^2}$ and $\frac{z^2}{c^2}$, once for $\frac{x^2}{a^2}$ and $\frac{z^2}{c^2}$, and once for $\frac{x^2}{a^2}$ and $\frac{y^2}{b^2}$) we get:

$$3\frac{x^2}{a^2} = 1 \qquad 3\frac{y^2}{b^2} = 1 \qquad 3\frac{z^2}{c^2} = 1$$

that is:

$$x = \frac{a}{\sqrt{3}} \qquad y = \frac{b}{\sqrt{3}} \qquad z = \frac{c}{\sqrt{3}}$$

Again, from the symmetry it is obvious that x, y, z are just half the sides of the parallelepiped and hence the lengths of the sides of the parallelepiped are $\frac{2a}{\sqrt{3}}$ (along the x axis), $\frac{2b}{\sqrt{3}}$ (along the y axis), and $\frac{2c}{\sqrt{3}}$ (along the z axis). Accordingly, the optimal volume of the parallelepiped is $\frac{2a}{\sqrt{3}} \times \frac{2b}{\sqrt{3}} \times \frac{2c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}$ (which can also be obtained from $8xyz$ which we used above).

Note: the optimal volume in this Problem is a maximum because the volume approaches zero when one or two of the sides of the parallelepiped approach zero.

2. What is the shape of the rectangular parallelepiped of optimal surface area that is inscribed inside a spheroid (i.e. ellipsoid of revolution) of a fixed shape?

   Answer: The equation of spheroid in standard form can be written as:

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1 \qquad (105)$$

If we follow in our footsteps in Problem 1 then we are required to optimize $h = f + \lambda g = (xy + xz + yz) + \lambda\left(\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2}\right)$ and hence:

$$\begin{aligned} \frac{\partial h}{\partial x} &= y + z + \frac{2\lambda x}{a^2} = 0 \\ \frac{\partial h}{\partial y} &= x + z + \frac{2\lambda y}{b^2} = 0 \\ \frac{\partial h}{\partial z} &= x + y + \frac{2\lambda z}{b^2} = 0 \end{aligned}$$

Now, from the symmetry we should have $z = y$ and hence the last three equations will reduce to only two, that is:

$$2y + \frac{2\lambda x}{a^2} = 0 \qquad (106)$$

$$x + y + \frac{2\lambda y}{b^2} = 0 \tag{107}$$

On solving Eq. 106 for λ we get $\lambda = -\frac{a^2 y}{x}$ and on substituting this in Eq. 107 we get:

$$\begin{aligned} x + y - \frac{2a^2y^2}{b^2x} &= 0 \\ x^2 + yx - \frac{2a^2y^2}{b^2} &= 0 \end{aligned}$$

On solving the last equation for x (using the quadratic formula) we get:

$$x = \frac{-y + \sqrt{y^2 + \frac{8a^2y^2}{b^2}}}{2} = \left(\frac{-1 + \sqrt{1 + \frac{8a^2}{b^2}}}{2}\right) y$$

where we ignored the non-sensible negative root. Now, on substituting from the last equation (plus $z = y$) into Eq. 105 we get:

$$\begin{aligned} \frac{1}{a^2}\left(\frac{-1 + \sqrt{1 + \frac{8a^2}{b^2}}}{2}\right)^2 y^2 + \frac{y^2 + y^2}{b^2} &= 1 \\ \left[\frac{1}{a^2}\left(\frac{-1 + \sqrt{1 + \frac{8a^2}{b^2}}}{2}\right)^2 + \frac{2}{b^2}\right] y^2 &= 1 \\ y \;=\; z &= \left[\frac{1}{a^2}\left(\frac{-1 + \sqrt{1 + \frac{8a^2}{b^2}}}{2}\right)^2 + \frac{2}{b^2}\right]^{-1/2} \end{aligned}$$

Accordingly, the inscribed parallelepiped of optimal surface area has dimensions:

$$\begin{aligned} 2x &= 2\left(\frac{-1 + \sqrt{1 + \frac{8a^2}{b^2}}}{2}\right)\left[\frac{1}{a^2}\left(\frac{-1 + \sqrt{1 + \frac{8a^2}{b^2}}}{2}\right)^2 + \frac{2}{b^2}\right]^{-1/2} \\ 2y \;=\; 2z &= 2\left[\frac{1}{a^2}\left(\frac{-1 + \sqrt{1 + \frac{8a^2}{b^2}}}{2}\right)^2 + \frac{2}{b^2}\right]^{-1/2} \end{aligned}$$

and its optimal surface area is $\sigma = 8(xy + xz + yz)$ where x, y, z are as given above.
Note: the optimal surface area in this Problem is a maximum because the surface area can approach zero when two sides of the parallelepiped (i.e. those corresponding to the identical semi-axes) approach zero.

3. A spheroid is given by the equation $x^2 + 3y^2 + 3z^2 = 1$. Find the dimensions and the surface area of the inscribed parallelepiped of optimal surface area.
Answer: On comparing this equation with Eq. 105 we get $a^2 = 1$ and $b^2 = 1/3$. Therefore, the dimensions and the surface area of the optimal parallelepiped are:

$$2x = 2\left(\frac{-1 + \sqrt{1 + \frac{8}{1/3}}}{2}\right)\left[\frac{1}{1}\left(\frac{-1 + \sqrt{1 + \frac{8}{1/3}}}{2}\right)^2 + \frac{2}{1/3}\right]^{-1/2} = \frac{4}{\sqrt{10}}$$

$$
\begin{aligned}
2y \;=\; 2z \;&=\; 2\left[\frac{1}{1}\left(\frac{-1+\sqrt{1+\frac{8}{1/3}}}{2}\right)^2+\frac{2}{1/3}\right]^{-1/2} = \frac{2}{\sqrt{10}} \\
\sigma \;&=\; 8\left(\frac{2}{\sqrt{10}}\frac{1}{\sqrt{10}}+\frac{2}{\sqrt{10}}\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{10}}\frac{1}{\sqrt{10}}\right) = 4
\end{aligned}
$$

4. What is the shape of the right circular cylinder of optimal surface area that is inscribed inside a sphere of a fixed radius?
Answer: Let the radius of the sphere be R while the radius and height of the cylinder be r and h (see Figure 44). From the Pythagoras theorem we have $h = 2\sqrt{R^2 - r^2}$ and hence the surface area of the cylinder is:

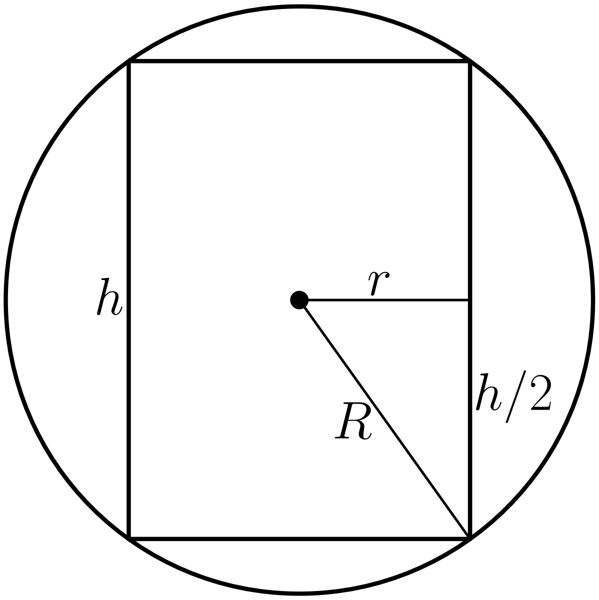

Figure 44: A cross section of a sphere of radius R in which a right circular cylinder of radius r and height h is inscribed. The cross section is through the center of the sphere (and the center of the cylinder) and along two generators of the cylinder. See Problem 4 of § 4.4.

$$\sigma = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 4\pi r\sqrt{R^2 - r^2} = 2\pi\left(r^2 + 2\sqrt{R^2r^2 - r^4}\right)$$

To optimize σ we take its derivative with respect to r (which is the optimization variable since R is fixed) and set it to zero to obtain the optimization condition, that is:

$$
\begin{aligned}
\frac{1}{2\pi}\frac{d\sigma}{dr} &= 0 \\
2r + 2\frac{2R^2r - 4r^3}{2\sqrt{R^2r^2 - r^4}} &= 0 \\
r + \frac{R^2r - 2r^3}{\sqrt{R^2r^2 - r^4}} &= 0 \\
\frac{\left(R^2r - 2r^3\right)^2}{R^2r^2 - r^4} &= r^2 \\
R^4r^2 - 4R^2r^4 + 4r^6 &= R^2r^4 - r^6 \\
5r^6 - 5R^2r^4 + R^4r^2 &= 0
\end{aligned}
$$

$$5r^4 - 5R^2r^2 + R^4 \quad = \quad 0 \qquad\qquad (r \neq 0)$$

On solving this equation for r^2 (using the quadratic formula) we get:

$$r^2 = \frac{5R^2 \pm \sqrt{25R^4 - 20R^4}}{10} = \left(\frac{5 \pm \sqrt{5}}{10}\right) R^2 \qquad \text{and hence} \qquad r = \left(\frac{5 \pm \sqrt{5}}{10}\right)^{1/2} R$$

Therefore:

$$h = 2\sqrt{R^2 - r^2} = 2\sqrt{R^2 - \left(\frac{5 \pm \sqrt{5}}{10}\right) R^2} = 2R\left(\frac{5 \mp \sqrt{5}}{10}\right)^{1/2}$$

On inserting the obtained expressions for r and h in the equation of the area (i.e. $\sigma = 2\pi r^2 + 2\pi rh$) we get:

$$\begin{aligned}
\sigma &= 2\pi\left(\frac{5 \pm \sqrt{5}}{10}\right) R^2 + 2\pi\left(\frac{5 \pm \sqrt{5}}{10}\right)^{1/2} R\left[2R\left(\frac{5 \mp \sqrt{5}}{10}\right)^{1/2}\right] \\
&= 2\pi R^2\left(\frac{5 \pm \sqrt{5}}{10}\right) + 4\pi R^2\left(\frac{5 \pm \sqrt{5}}{10}\right)^{1/2}\left(\frac{5 \mp \sqrt{5}}{10}\right)^{1/2} \\
&= 2\pi R^2\left[\left(\frac{5 \pm \sqrt{5}}{10}\right) + 2\left(\frac{5^2 - 5}{100}\right)^{1/2}\right] \\
&= 2\pi R^2\left[\left(\frac{5 \pm \sqrt{5}}{10}\right) + \frac{\sqrt{20}}{5}\right]
\end{aligned}$$

Now, since the optimal surface area is a maximum (see the upcoming notes), we take $r = \left(\frac{5+\sqrt{5}}{10}\right)^{1/2} R$ and $h = 2R\left(\frac{5-\sqrt{5}}{10}\right)^{1/2}$ and hence the optimal surface area is:

$$\sigma = 2\pi R^2\left[\left(\frac{5 + \sqrt{5}}{10}\right) + \frac{\sqrt{20}}{5}\right] = \pi\left(1 + \sqrt{5}\right) R^2 \simeq 10.1664R^2$$

This is less than the surface area of the sphere which is $4\pi R^2 \simeq 12.5664R^2$ (as it should be).
Note 1: the optimal surface area in this Problem is a maximum because σ can approach zero when r approaches zero (and h approaches $2R$).[89]
Note 2: for more clarity, we plotted in Figure 45 the surface area σ of a cylinder inscribed inside a unit sphere as a function of the cylinder radius r.[90]

5. What is the shape of the right circular cylinder of optimal volume that is inscribed inside a sphere of a fixed radius?
Answer: If we follow in our footsteps in Problem 4 then the volume V of the cylinder is given by:

$$V = \pi r^2 h = \pi h\left(R^2 - \frac{h^2}{4}\right) = \pi\left(R^2 h - \frac{h^3}{4}\right)$$

To optimize V we take its derivative with respect to h (which is the optimization variable since R is fixed) and set it to zero to obtain the optimization condition, that is:

$$\frac{1}{\pi}\frac{dV}{dh} \quad = \quad 0$$

---

[89] We should also note that the above optimal area $\left[\text{i.e. } \sigma = \pi(1+\sqrt{5})R^2\right]$ is also larger than the area in the other limiting case when h approaches zero and hence the surface area approaches $2\pi R^2$.

[90] The use of unit sphere does not affect the generality because this is equivalent to using length units scaled by $1/R$ and using area units scaled by $1/R^2$.

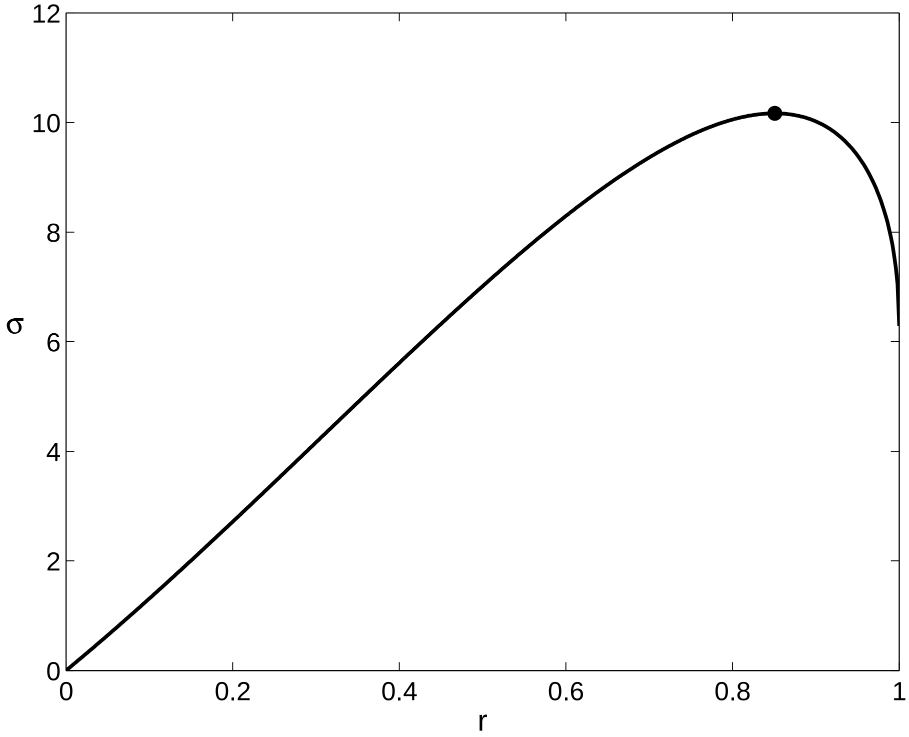

Figure 45: Plot of the surface area σ (of a cylinder inscribed inside a unit sphere) as a function of the cylinder radius r where the peak of the curve at $(\sqrt{\frac{5+\sqrt{5}}{10}}, \pi + \pi\sqrt{5}) \simeq (0.8507, 10.1664)$ is marked. See Problem 4 of § 4.4.

$$\begin{aligned} R^2 - \frac{3h^2}{4} &= 0 \\ h &= \frac{2R}{\sqrt{3}} \end{aligned}$$

Hence, the optimal cylinder has a height $h = \frac{2R}{\sqrt{3}}$ and a radius $r = \sqrt{R^2 - \frac{h^2}{4}} = \sqrt{R^2 - \frac{R^2}{3}} = R\sqrt{\frac{2}{3}}$ with an optimal volume $V = \frac{4\pi}{3\sqrt{3}}R^3$.

Note: the optimal volume in this Problem is a maximum because V can approach zero when r approaches zero (and h approaches $2R$) or when h approaches zero (and r approaches R).

6. What is the shape of the right circular cylinder of optimal volume that is inscribed inside a right circular cone of a fixed shape?

   Answer: Let r and h be the radius and height of the cylinder and R and H the base radius and height of the cone (see Figure 46). Now, the triangles ABC and ADE are similar and hence:

$$\frac{H-h}{H} = \frac{r}{R} \qquad \rightarrow \qquad h = H\left(1 - \frac{r}{R}\right)$$

   Thus, the volume of the cylinder is:

$$V = \pi r^2 h = \pi r^2 H\left(1 - \frac{r}{R}\right) = \pi H\left(r^2 - \frac{r^3}{R}\right)$$

   To optimize V we take its derivative with respect to r (which is the optimization variable since R and

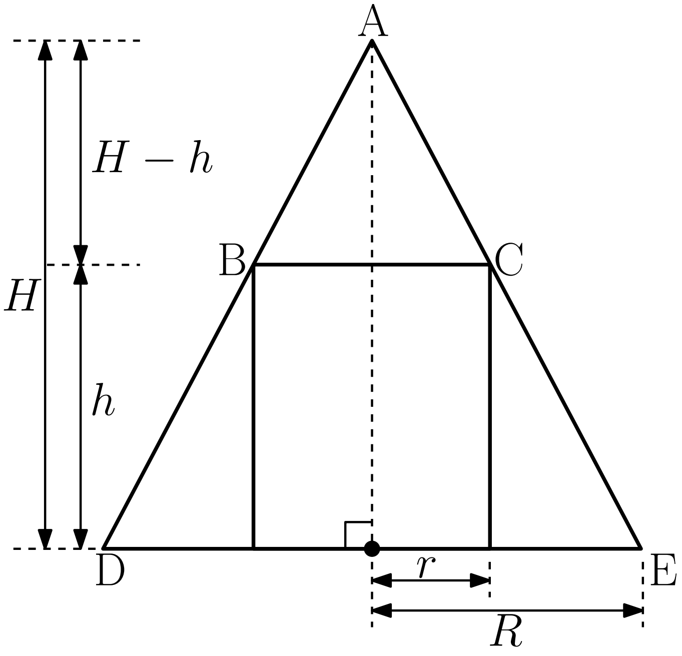

Figure 46: A cross section of a right circular cylinder of radius r and height h inscribed inside a right circular cone of base radius R and height H. The cross section is through the apex A of the cone and the diameter of its base (and hence through the center of the cylinder and along two of its generators). See Problem 6 of § 4.4.

H are fixed) and set it to zero to obtain the optimization condition, that is:

$$\begin{aligned} \frac{1}{\pi}\frac{dV}{dr} &= 0 \\ H\left(2r - 3\frac{r^2}{R}\right) &= 0 \\ 2 - 3\frac{r}{R} &= 0 \qquad (H \neq 0,\ r \neq 0) \\ r &= \frac{2}{3}R \end{aligned}$$

Thus, the cylinder of optimal volume has $r = \frac{2}{3}R$ and $h = H\left(1 - \frac{r}{R}\right) = H\left(1 - \frac{1}{R}\frac{2}{3}R\right) = \frac{H}{3}$ with an optimal volume $V = \pi(\frac{2}{3}R)^2\frac{H}{3} = \frac{4\pi}{27}R^2H$.
Note: the optimal volume in this Problem is a maximum because V can approach zero when h approaches H (or when r approaches R).

7. What is the shape of the right circular cone of optimal volume that is inscribed inside a sphere of a fixed radius?
   Answer: Let R be the radius of the sphere and r and h be the base radius and height of the cone (see Figure 47). Now, $r = R\cos\theta$ and $h = R + R\sin\theta$ and hence the volume of the cone is:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(R^2\cos^2\theta\right)(R + R\sin\theta) = \frac{\pi R^3}{3}\left(\cos^2\theta + \cos^2\theta\sin\theta\right)$$

   To optimize V we take its derivative with respect to θ (which is the optimization variable since R is fixed) and set it to zero to obtain the optimization condition, that is:

$$\frac{3}{\pi R^3}\frac{dV}{d\theta} = 0$$

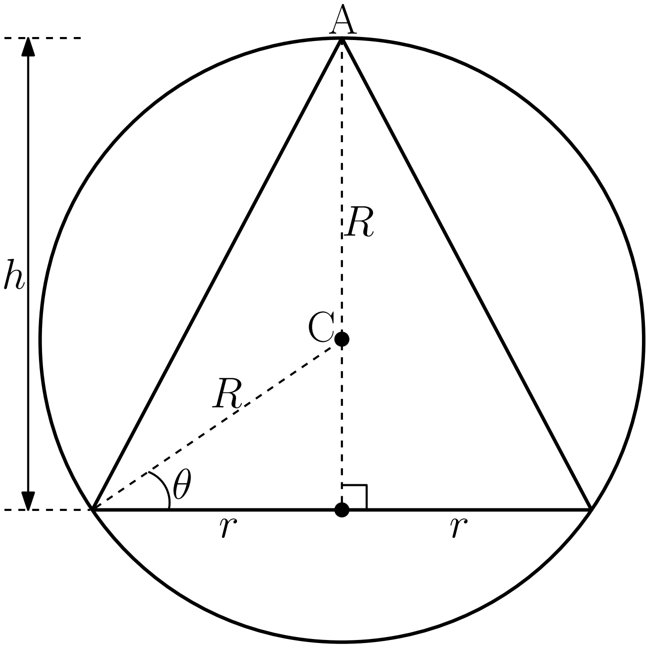

Figure 47: A cross section of a sphere of radius R inside which a right circular cone of base radius r and height h is inscribed. The cross section is through the apex A of the cone and the diameter of its base (and hence through the center C of the sphere). See Problem 7 of § 4.4.

$$\begin{aligned}
-2\cos\theta\sin\theta-2\cos\theta\sin^2\theta+\cos^3\theta &= 0\\
-2\sin\theta-2\sin^2\theta+\cos^2\theta &= 0 \qquad (\cos\theta\neq 0 \text{ since } r\neq 0)\\
-2\sin\theta-3\sin^2\theta+\cos^2\theta+\sin^2\theta &= 0\\
-2\sin\theta-3\sin^2\theta+1 &= 0\\
3\sin^2\theta+2\sin\theta-1 &= 0\\
(3\sin\theta-1)(\sin\theta+1) &= 0
\end{aligned}$$

Hence, $\sin\theta=\frac{1}{3}$ because $\sin\theta=-1$ is unacceptable (since $r\neq 0$). Accordingly, $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\frac{1}{9}}=\frac{2\sqrt{2}}{3}$ and hence the optimal cone has $r=\frac{2\sqrt{2}}{3}R$ and $h=R+\frac{R}{3}=\frac{4}{3}R$ with an optimal volume $V=\frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)=\frac{32}{81}\pi R^3$.

Note: the optimal volume in this Problem is a maximum because V can converge to zero when h approaches $2R$ (or r approaches zero).

8. What is the shape of the right circular cone of optimal volume inside which a sphere of fixed radius is inscribed?

   Answer: Let the radius of the sphere be r and the base radius and height of the cone be R and H (see Figure 48). Now, the triangles ABC and ADE are similar (since they are right triangles sharing an angle) and hence:

$$\begin{aligned}
\frac{r}{R} &= \frac{\sqrt{(H-r)^2-r^2}}{H}\\
\frac{r}{R} &= \frac{\sqrt{H^2-2rH}}{H}\\
\frac{r^2}{R^2} &= \frac{H^2-2rH}{H^2}
\end{aligned}$$

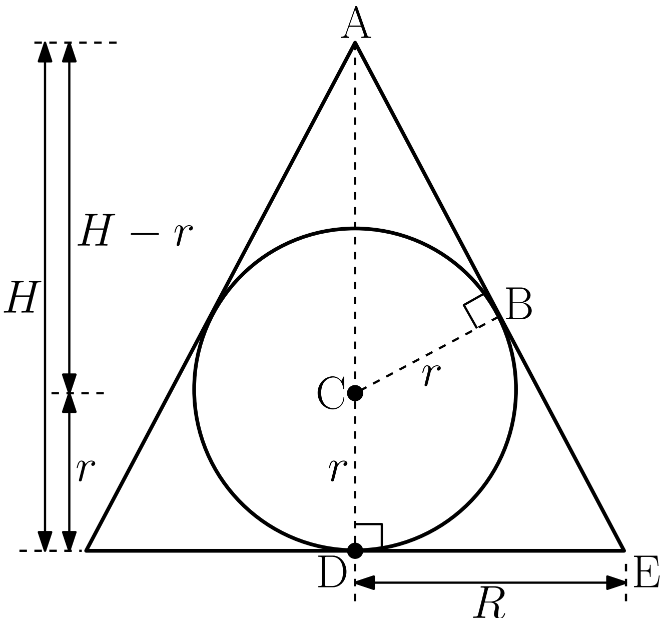

Figure 48: A cross section of a sphere of radius r inscribed inside a right circular cone of base radius R and height H. The cross section is through the apex A of the cone and the diameter of its base (and hence through the center C of the circle). See Problem 8 of § 4.4.

$$\begin{aligned} r^2H^2 &= R^2H^2 - 2rR^2H \\ \left(R^2 - r^2\right)H^2 &= 2rR^2H \\ \left(R^2 - r^2\right)H &= 2rR^2 \qquad\qquad (H \neq 0) \\ H &= \frac{2rR^2}{R^2 - r^2} \end{aligned}$$

Now, the volume of the cone is:

$$V = \frac{1}{3}\pi R^2 H = \frac{1}{3}\pi R^2 \left(\frac{2rR^2}{R^2 - r^2}\right) = \frac{2\pi r R^4}{3\left(R^2 - r^2\right)}$$

To optimize V we take its derivative with respect to R (which is the optimization variable since r is fixed) and set it to zero to obtain the optimization condition, that is:

$$\begin{aligned} \frac{3}{2\pi}\frac{dV}{dR} &= 0 \\ \frac{4rR^3}{\left(R^2 - r^2\right)} - \frac{rR^4\,(2R)}{\left(R^2 - r^2\right)^2} &= 0 \\ \frac{4rR^3\left(R^2 - r^2\right)}{\left(R^2 - r^2\right)^2} - \frac{2rR^5}{\left(R^2 - r^2\right)^2} &= 0 \\ 4rR^5 - 4r^3R^3 - 2rR^5 &= 0 \\ 2rR^5 - 4r^3R^3 &= 0 \\ R^2 - 2r^2 &= 0 \qquad\qquad (r \neq 0,\ R \neq 0) \\ R &= \sqrt{2}r \end{aligned}$$

So, the optimal cone has $R = \sqrt{2}r$ and $H = \frac{2r\left(\sqrt{2}r\right)^2}{\left(\sqrt{2}r\right)^2 - r^2} = 4r$ with an optimal volume $V = \frac{1}{3}\pi\left(\sqrt{2}r\right)^2(4r) = \frac{8}{3}\pi r^3$.

Note 1: the optimal volume in this Problem is a minimum because V can diverge when H approaches $2r$ (or when R approaches r).
Note 2: for more clarity, we plotted in Figure 49 the volume V of a cone (in which a unit sphere is inscribed) as a function of the cone radius R.[91]

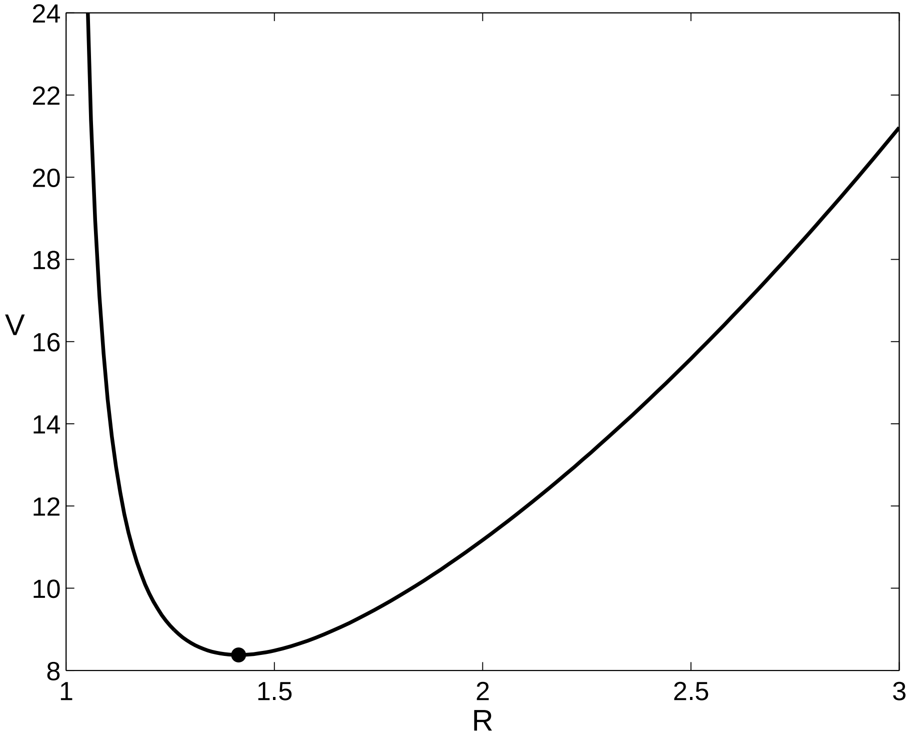

Figure 49: Plot of the volume V (of a right circular cone in which a unit sphere is inscribed) as a function of the cone base radius R where the minimum of the curve at $(\sqrt{2}, \frac{8\pi}{3}) \simeq (1.4142, 8.3776)$ is marked. See Problem 8 of § 4.4.

4.5 Solids of Revolution of Optimal Volume

From the title of this section, it is clear that in this type of problems it is required to optimize the volume of a solid of revolution generated by revolving a certain curve $\big[$which is usually represented by a function $y = y(x)\big]$ around a straight line (which is usually the x axis). As we will see, some of these problems can be solved by ordinary calculus without need for the variational formulation of the Euler-Lagrange equation.

Problems

1. Find and plot the parabola $y(x)$ that passes through the points $(0, 0)$ and $(1, 1)$ and optimizes the volume generated by its revolution around the x axis.
   Answer: The equation of parabola is $y = ax^2 + bx + c$ where a, b, c are constants and $a \neq 0$. Now, since it passes through $(0, 0)$ then $c = 0$, and since it passes through $(1, 1)$ then $a + b = 1$ and hence $b = 1 - a$. Therefore, the parabola should be given by $y = ax^2 + (1 - a)x$. Now, the volume generated by the revolution of this parabola around the x axis between $(0, 0)$ and $(1, 1)$ is given by:

$$V = \int_0^1 \pi y^2 \, dx = \pi \int_0^1 \left[ax^2 + (1 - a)\, x\right]^2 \, dx$$

---

[91] Again, the use of unit sphere does not affect the generality because it is a scaling issue.

So, if the volume is optimum then it should be stationary with respect to variations in the parameter a of the parabola (which controls the shape of the parabola and hence the volume), that is:

$$\begin{aligned}
\frac{1}{\pi}\frac{dV}{da} &= 0 \\
\frac{d}{da}\left\{\int_0^1 \left[ax^2+(1-a)x\right]^2 dx\right\} &= 0 && (108)\\
\int_0^1 \left\{\frac{\partial}{\partial a}\left[ax^2+(1-a)x\right]^2\right\} dx &= 0 && (109)\\
\int_0^1 \left\{2\left[ax^2+(1-a)x\right]\left[x^2-x\right]\right\} dx &= 0 \\
2\int_0^1 \left[ax^4+(1-a)x^3-ax^3-(1-a)x^2\right] dx &= 0 \\
\int_0^1 \left[ax^4+(1-2a)x^3-(1-a)x^2\right] dx &= 0 \\
\left[\frac{a}{5}x^5+\frac{(1-2a)}{4}x^4-\frac{(1-a)}{3}x^3\right]_0^1 &= 0 \\
\left[\frac{a}{5}+\frac{(1-2a)}{4}-\frac{(1-a)}{3}\right]-[0+0-0] &= 0 \\
\frac{12a+15-30a-20+20a}{60} &= 0 \\
\frac{2a-5}{60} &= 0 \\
a &= \frac{5}{2}
\end{aligned}$$

Therefore, the volume-optimizing parabola is given by:

$$y=\frac{5}{2}x^2+\left(1-\frac{5}{2}\right)x=\frac{5}{2}x^2-\frac{3}{2}x$$

This parabola is plotted in Figure 50.

Note 1: regarding the differentiation of the integral (see Eqs. 108 and 109) we are using the fact that if f is a function of two variables (t and x) and

$$\phi(t)=\int_\alpha^\beta f(t,x)\,dx$$

(with α and β being constants) then

$$\frac{d\phi(t)}{dt}=\int_\alpha^\beta \frac{\partial f(t,x)}{\partial t}\,dx$$

In fact, we do not need this complication (i.e. differentiating the integral) because we can integrate first (without differentiation) to obtain an expression for the volume as a function of a, that is:

$$V=\pi\left(\frac{a^2-5a+10}{30}\right) \qquad \text{(see Problem 2)}$$

This expression can then be differentiated with respect to a and set to zero, that is:

$$\frac{dV}{da}=\pi\left(\frac{2a-5}{30}\right)=0 \qquad \text{and hence} \qquad 2a-5=0$$

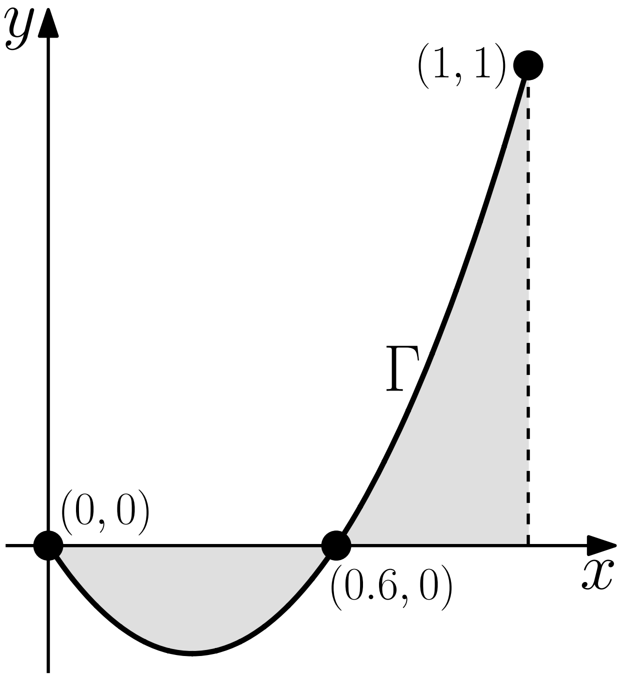

Figure 50: Plot of the parabola Γ that passes through the points $(0,0)$ and $(1,1)$ and optimizes the volume generated by its revolution around the x axis. See Problem 1 of § 4.5.

which leads to $a = 5/2$ (as before). However, we followed this method (which requires differentiating the integral) for diversity and practice.[92]

Note 2: although the immediate objective here is to find a curve, the ultimate objective is to find the shape of a 3D solid object (of optimal volume) and hence the Problem is classified as an optimal solid issue. The sufficiency of finding a curve is due to the fact that we are dealing with a solid of revolution which is completely defined by its profile curve (plus the x axis).

2. Show that the solution that we obtained in Problem 1 is a minimum.

   Answer: It should be intuitively obvious that the solution in Problem 1 is a minimum because for some parabolas (passing through the designated boundary points) the volume can diverge. However, we can show this formally by calculating the volume (as a function of a) and plotting it, that is:

$$\begin{aligned}
V &= \pi \int_0^1 \left[ax^2 + (1-a)\,x\right]^2 \, dx \\
&= \pi \int_0^1 \left[a^2x^4 + 2ax^3 - 2a^2x^3 + x^2 - 2ax^2 + a^2x^2\right] \, dx \\
&= \pi \int_0^1 \left[a^2x^4 + \left(2a - 2a^2\right) x^3 + \left(1 - 2a + a^2\right) x^2\right] \, dx \\
&= \pi \left[\frac{a^2}{5}x^5 + \left(\frac{2a-2a^2}{4}\right) x^4 + \left(\frac{1-2a+a^2}{3}\right) x^3\right]_0^1 \\
&= \pi \left[\frac{a^2}{5} + \left(\frac{2a-2a^2}{4}\right) + \left(\frac{1-2a+a^2}{3}\right)\right] \\
&= \pi \left(\frac{6a^2}{30} + \frac{15a - 15a^2}{30} + \frac{10 - 20a + 10a^2}{30}\right)
\end{aligned}$$

---

[92] In fact, this method may also reduce the required algebra and can help in simplifying the integration (which can be essential in some cases).

$$= \pi \left(\frac{a^2 - 5a + 10}{30} \right)$$

On plotting the volume as a function of a (see Figure 51) we can see clearly that the volume has a minimum ($V = \pi/8$) at $a = 5/2$ and hence the solution is a minimum (rather than a maximum or inflection).

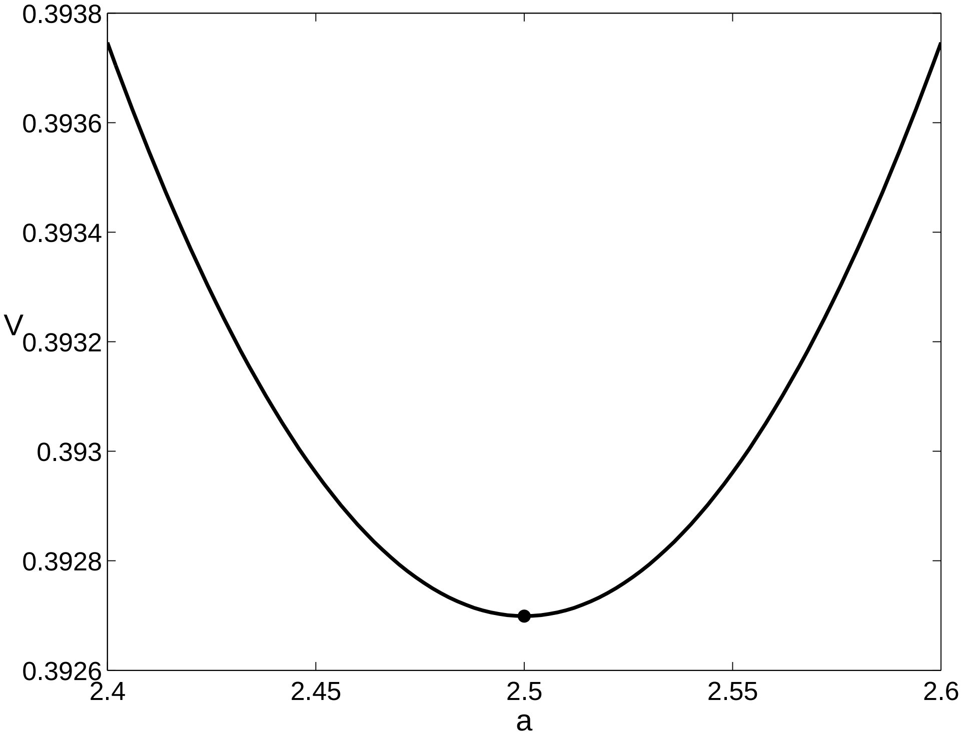

Figure 51: Plot of the volume V of the parabola of Problem 2 of § 4.5 as a function of a where the lowest point $(5/2, \pi/8)$ is marked.

3. Find the cubic curve $y(x)$ that passes through the points $(0, 0)$ and $(1, 1)$ and optimizes the volume generated by its revolution around the x axis. Also, plot the curve between $x = 0$ and $x = 1$.
Answer: The equation of cubic curve is $y = ax^3 + bx^2 + cx + d$ where a, b, c, d are constants and $a \neq 0$. Now, since the curve passes through $(0, 0)$ then $d = 0$, and since it passes through $(1, 1)$ then $a+b+c = 1$ and hence $c = 1 - a - b$. Therefore, the curve should be given by $y = ax^3 + bx^2 + (1 - a - b)\,x$. Now, the volume generated by the revolution of this cubic curve around the x axis between $(0, 0)$ and $(1, 1)$ is given by:

$$\begin{aligned} V &= \int_0^1 \pi y^2 \, dx \\ &= \pi \int_0^1 \left[ax^3 + bx^2 + (1 - a - b)\,x \right]^2 \, dx \\ &= \pi \int_0^1 \Big[a^2 x^6 + 2abx^5 + \left(-2a^2 - 2ab + b^2 + 2a \right) x^4 + \\ &\qquad \left(-2ab - 2b^2 + 2b \right) x^3 + \left(a^2 + 2ab + b^2 - 2a - 2b + 1 \right) x^2 \Big] \, dx \end{aligned}$$

$$\begin{aligned}
&= \pi\left[\frac{a^2}{7}x^7+\frac{2ab}{6}x^6+\left(\frac{-2a^2-2ab+b^2+2a}{5}\right)x^5+\right.\\
&\qquad\left.\left(\frac{-2ab-2b^2+2b}{4}\right)x^4+\left(\frac{a^2+2ab+b^2-2a-2b+1}{3}\right)x^3\right]_0^1\\
&= \pi\left[\frac{a^2}{7}+\frac{2ab}{6}+\left(\frac{-2a^2-2ab+b^2+2a}{5}\right)+\left(\frac{-2ab-2b^2+2b}{4}\right)+\right.\\
&\qquad\left.\left(\frac{a^2+2ab+b^2-2a-2b+1}{3}\right)\right]-0\\
&= \pi\left(\frac{8a^2}{105}+\frac{ab}{10}+\frac{b^2}{30}-\frac{4a}{15}-\frac{b}{6}+\frac{1}{3}\right) \qquad (110)
\end{aligned}$$

Now, if the volume is optimum then it should be stationary with respect to variations in the parameters a and b of the curve (since these parameters control the shape of the curve and hence the volume), that is:

$$\begin{aligned}
\frac{1}{\pi}\frac{\partial V}{\partial a} &= \frac{16}{105}a+\frac{1}{10}b-\frac{4}{15}=0\\
\frac{1}{\pi}\frac{\partial V}{\partial b} &= \frac{1}{10}a+\frac{1}{15}b-\frac{1}{6}=0
\end{aligned}$$

On solving this system of simultaneous equations we obtain $a = 7$ and $b = -8$ and hence $c = 1-7+8 = 2$. Accordingly, the cubic curve is represented by the function $y = 7x^3 - 8x^2 + 2x$ and it is plotted in Figure 52.

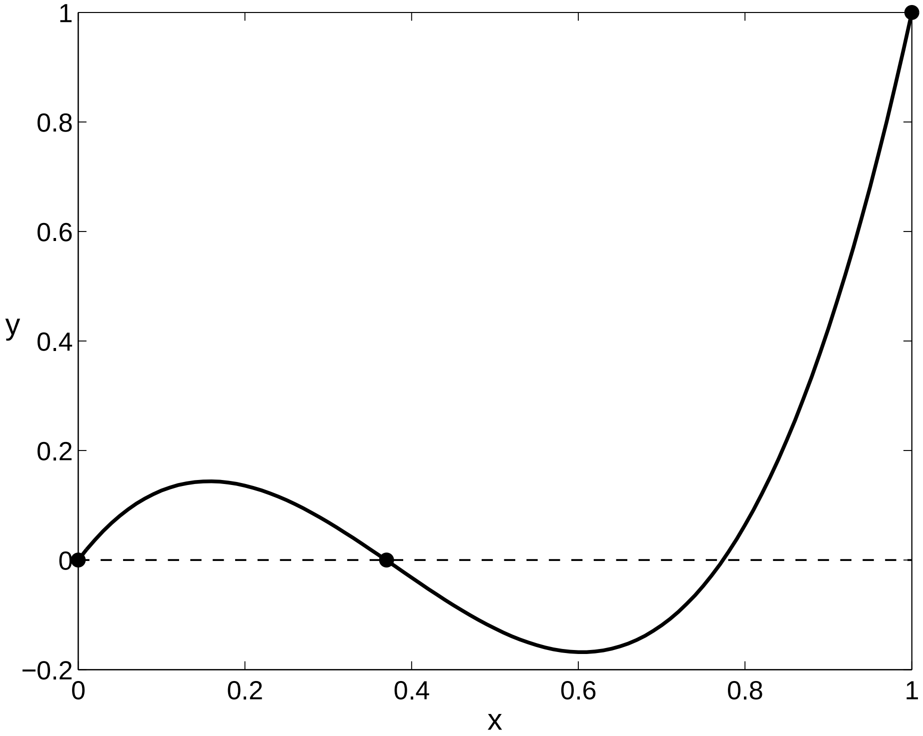

Figure 52: Plot of the cubic curve $y = 7x^3 - 8x^2 + 2x$ between $x = 0$ and $x = 1$. See Problem 3 of § 4.5.

Table 2: A table for the values of V as a function of a and b where a minimum (for V) can be seen at $(a, b) = (7, -8)$. Refer to Problem 4 of § 4.5 for details.

		b						
		-8.03	**-8.02**	**-8.01**	**-8.00**	**-7.99**	**-7.98**	**-7.97**
a	**6.96**	0.2102937	0.2101157	0.2099586	0.2098225	0.2097073	0.2096130	0.2095397
	6.97	0.2100319	0.2098853	0.2097597	0.2096549	0.2095712	0.2095083	0.2094664
	6.98	0.2098180	0.2097028	0.2096086	0.2095353	0.2094829	0.2094515	0.2094410
	6.99	0.2096519	0.2095682	0.2095053	0.2094634	0.2094425	0.2094425	0.2094634
	7.00	0.2095338	0.2094814	0.2094500	**0.2094395**	0.2094500	0.2094814	0.2095338
	7.01	0.2094634	0.2094425	0.2094425	0.2094634	0.2095053	0.2095682	0.2096519
	7.02	0.2094410	0.2094515	0.2094829	0.2095353	0.2096086	0.2097028	0.2098180
	7.03	0.2094664	0.2095083	0.2095712	0.2096549	0.2097597	0.2098853	0.2100319
	7.04	0.2095397	0.2096130	0.2097073	0.2098225	0.2099586	0.2101157	0.2102937

4. Show that the solution that we obtained in Problem 3 is a minimum.
   Answer: It should be intuitively obvious that the solution in Problem 3 is a minimum because for some cubic curves (passing through the designated boundary points) the volume can diverge. However, we can show this formally by calculating the volume (as a function of a and b as given by Eq. 110) around $(a, b) = (7, -8)$ where we can see (refer to Table 2) that the point $(a, b, V) = (7, -8, 0.2094395)$ is a minimum.
5. Find the shape of the closed solid of revolution of fixed surface area and optimal volume.[93]
   Answer: Because the surface is closed we can (without loss of generality) represent the generating curve Γ by a function $y(x)$ with its two end points (A and B) being on the x axis (and hence the surface is generated by rotating Γ around the x axis). We can also (without loss of generality) put the point A at the origin of coordinates. So, the setting of this Problem can be depicted schematically as in Figure 53.
   Our objective is to optimize the volume of this solid subject to the constraint on its surface area and hence we use the Lagrange multipliers technique (see § 1.8). Now, the volume is given by the integral $\int_0^{x_B} \pi y^2\, dx$ while the surface area is given by the integral $\int_\Gamma 2\pi y\, ds = \int_0^{x_B} 2\pi y\sqrt{1 + y'^2}\, dx$. So, $F = \pi y^2$ and $G = 2\pi y\sqrt{1 + y'^2}$ and hence $H \equiv F + \lambda G = \pi y^2 + \lambda 2\pi y\sqrt{1 + y'^2}$. Using Beltrami identity (i.e. Eq. 3 with H replacing F), we have:

$$\begin{aligned}
H - y'\frac{\partial H}{\partial y'} &= C \\
\left(\pi y^2 + \lambda 2\pi y\sqrt{1 + y'^2}\right) - y'\frac{\partial}{\partial y'}\left(\pi y^2 + \lambda 2\pi y\sqrt{1 + y'^2}\right) &= C \\
\pi y^2 + \lambda 2\pi y\sqrt{1 + y'^2} - y'\left(\lambda 2\pi y\frac{2y'}{2\sqrt{1 + y'^2}}\right) &= C \\
\pi y^2 + \lambda 2\pi y\sqrt{1 + y'^2} - \frac{\lambda 2\pi y y'^2}{\sqrt{1 + y'^2}} &= C \\
\pi y^2 + \frac{\lambda 2\pi y\left(1 + y'^2\right) - \lambda 2\pi y y'^2}{\sqrt{1 + y'^2}} &= C \\
\pi y^2 + \frac{\lambda 2\pi y}{\sqrt{1 + y'^2}} &= C
\end{aligned}$$

---

[93] "Closed" here means that the end points of the revolving curve that generates the surface are fixed in space and hence they do not rotate. In other words, the end points are on the axis of revolution (i.e. the line which the curve revolves around).

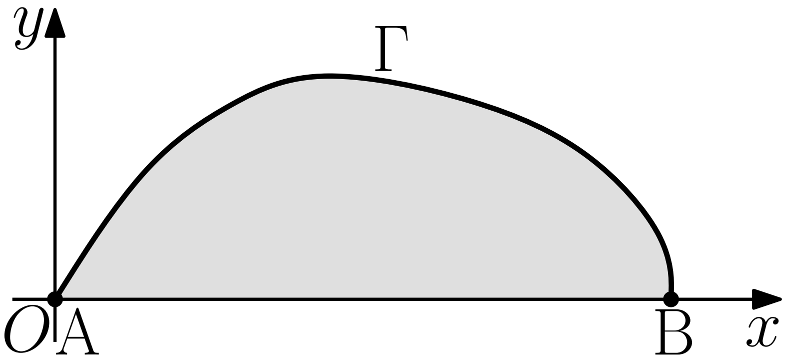

Figure 53: A simple sketch depicting the setting of Problem 5 of § 4.5 where the curve Γ revolves around the x axis to generate a closed solid of revolution.

Now, because the curve passes through the origin (where $y = 0$), C is zero. Hence:

$$\begin{aligned}
\pi y^2 + \frac{\lambda 2\pi y}{\sqrt{1+y'^2}} &= 0 \\
y + \frac{2\lambda}{\sqrt{1+y'^2}} &= 0 \\
\sqrt{1+y'^2} &= -\frac{2\lambda}{y} \\
y'^2 &= \frac{4\lambda^2}{y^2} - 1 \\
\frac{dy}{dx} &= \frac{\sqrt{4\lambda^2 - y^2}}{y} \\
\int \frac{y\,dy}{\sqrt{4\lambda^2 - y^2}} &= \int dx \\
-\sqrt{4\lambda^2 - y^2} &= x + C_1
\end{aligned}$$

Now, because the curve passes through the origin (where $x = 0$ and $y = 0$), $C_1 = -2\lambda$ and hence:

$$\begin{aligned}
-\sqrt{4\lambda^2 - y^2} &= x - 2\lambda \\
4\lambda^2 - y^2 &= (x - 2\lambda)^2 \\
(x - 2\lambda)^2 + y^2 &= 4\lambda^2
\end{aligned}$$

This is an equation of a circle with center $(2\lambda, 0)$ and radius 2λ. Therefore, the optimal solid of revolution is a sphere with center $(2\lambda, 0, z_0)$ and radius 2λ (with z_0 being a given number representing the z coordinate of the center).

Note 1: the optimal volume in this Problem is a maximum because V will converge to zero when the revolving curve (and hence the surface) approaches the axis of rotation (or when the two sides of the curve approach each other).

Note 2: because $y(x)$ is presumed to be a function over its domain, we exclude one-to-many curves. However, this will not affect the generality of the result noting that the optimal volume is a maximum (not a minimum).

Note 3: in any specific problem of this type, λ can be determined from the given area σ of the surface. For example, if the given area is 16π then we have:

$$\sigma = 16\pi$$

$$\begin{aligned} 4\pi\left(2\lambda\right)^2 &= 16\pi \\ 16\pi\lambda^2 &= 16\pi \\ \lambda &= 1 \end{aligned}$$

where in line 2 we used the formula for the area of sphere (i.e. $\sigma = 4\pi R^2$). Accordingly, the equation of the circle is $\left(x-2\right)^2 + y^2 = 4$ and the equation of the sphere is $\left(x-2\right)^2 + y^2 + (z-z_0)^2 = 4$.

4.6 Solids of Revolution of Optimal Surface Area

From the title of this section, it is clear that in this type of problems it is required to optimize the surface area of a solid of revolution generated by revolving a certain curve $\big[$which is usually represented by a function $y = y(x)\big]$ around a straight line (which is usually the x axis).

Problems

1. What is the shape of the solid of revolution of fixed volume and optimal (i.e. minimum) surface area?
   Answer: It should be sphere. This is a consequence (or corollary) of the result of Problem 5 of § 4.5 because if the solid of the given volume (say V_0) and minimum surface area (say σ_0) was not sphere then the sphere of the volume V_0 should have larger surface area than σ_0 and hence the sphere of surface area σ_0 should have smaller volume than V_0 and this contradicts the result of Problem 5 of § 4.5 because the sphere of surface area σ_0 should have maximum volume (in comparison to any other solid of revolution of the same surface area).[94]

4.7 Solids of Optimal Surface Area

In this type of problems it is required to optimize the surface area of solids of arbitrary shapes subject to certain condition(s) such as having fixed volume.

Problems

1. What is the shape of the solid of fixed volume and optimal (i.e. minimum) surface area?[95]
   Answer: This is an area optimization problem with a volume constraint. However, we do not need to use the Lagrange multipliers technique or the formalism of the calculus of variations directly. What we will use in this answer is an intuitive approach based on the use of the calculus of variations indirectly, i.e. as applied to Problems of lower dimensionality that we investigated earlier. This will be clarified in the following discussion.
   Let have a solid of arbitrary shape which has a given volume. Suppose that we divide this solid into infinitesimal slices each of width ds and arbitrary cross sectional shape. The volume of each one of these slices is obviously the product of the cross sectional area times the width ds. Now, according to Problem 11 of § 2.4 the shape of the closed plane curve of shortest length that encloses a given area is a circle and hence the slice can be replaced by a circular disc of width ds and of the same volume (and hence the same cross sectional area). Accordingly, the surface area of the circular disc (which is equal to its perimeter times ds) is smaller than the surface area of the original slice (because the perimeter of the disc is shorter while ds is common to both), and since the surface area is the sum of the surface areas of the slices then the surface area of the original solid is larger than (or equal to) the surface area of the stack of discs. Now, the stack of discs (assuming that they are centered on a common axis) is a surface of revolution whose volume is the same as the volume of the original solid while its surface area is less than (or equal to) the surface area of the original solid. Referring to Problem 1 of § 4.6 the solid of revolution of a given volume and minimum surface area is a sphere, and hence the surface area of the sphere of the given volume should be less than (or equal to) the surface area of the stack of

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[94] For sphere, $V = \frac{4}{3}\pi R^3$ and $\sigma = 4\pi R^2$ (where V is its volume, σ is its surface area, and R is its radius) which can be combined to obtain $V = \frac{4}{3}\pi\left(\frac{\sigma}{4\pi}\right)^{3/2}$ and hence the volume increases/decreases as the surface area increases/decreases (and vice versa).

[95] “Solid” here means 3D continuous object with no cavity.

discs. Accordingly, the surface area of the sphere of the same volume should be less than (or equal to) the surface area of the original solid, i.e. the surface area of the sphere is the minimum of all surface areas of all solids of the same volume. So, the shape of the solid of fixed volume and optimal (i.e. minimum) surface area is sphere.

Note: the optimal surface area in this Problem is a minimum because the surface area of some 3D shapes will diverge when one of the dimensions approaches zero (noting that the volume is fixed). This can also be concluded from the results of the Problems that we used in our answer of the present Problem.

2. Find the shape of the solid of minimum surface area with a volume $V = 36\pi$. Also, find the surface area of this solid.

   Answer: From the answer of Problem 1 the solid should be sphere. From the formula of the volume of sphere of radius R we have $V = \frac{4}{3}\pi R^3 = 36\pi$ and hence $R = 3$. Therefore, the solid is a sphere of radius $R = 3$. Also, from the formula of the surface area σ of sphere of radius R we have $\sigma = 4\pi R^2 = 4\pi \times 3^2 = 36\pi$. So, our solid is a sphere of surface area 36π (area units) and volume 36π (volume units).

4.8 Solids of Optimal Volume

In this type of problems it is required to optimize the volume of solids of arbitrary shapes subject to certain condition(s) such as having fixed surface area.

Problems

1. What is the shape of the solid of fixed surface area and optimal (i.e. maximum) volume?[96]

   Answer: This is a volume optimization problem with an area constraint. We can follow a similar approach to that used in Problem 1 of § 4.7 to show that the shape is a sphere. However, it is easier to use the result of Problem 1 of § 4.7 (in conjunction with the proof by contradiction method) to obtain this result because if the solid of the given surface area (say σ_0) and optimal (i.e. maximum) volume (say V_0) was not sphere then the sphere of the surface area σ_0 should have less volume than V_0 and hence the sphere of volume V_0 should have larger surface area than σ_0 and this contradicts the result of Problem 1 of § 4.7 because the sphere of volume V_0 should have minimum surface area (in comparison to any other solid of the same volume).[97]

   Note: the optimal volume in this Problem is a maximum because the volume of some 3D shapes will converge to zero when one of the dimensions approaches zero (noting that the surface area is fixed).

2. Find the shape of the solid of maximum volume with a surface area $\sigma = 16\pi$. Also, find the volume of this solid.

   Answer: From the answer of Problem 1 the solid should be sphere. From the formula of the surface area of sphere of radius R we have $\sigma = 4\pi R^2 = 16\pi$ and hence $R = 2$. Therefore, the solid is a sphere of radius $R = 2$. Also, from the formula of the volume V of sphere of radius R we have $V = \frac{4}{3}\pi R^3 = \frac{32}{3}\pi$. So, our solid is a sphere of surface area 16π (area units) and volume $\frac{32}{3}\pi$ (volume units).

4.9 Solids of Revolution of Optimal Resistance to Fluid Flow

In this section we present and solve some variational problems related to solids of revolution whose resistance to fluid flow is optimized.

Problems

1. Let have a solid of revolution that is totally submerged in a fluid and it is in a state of uniform motion relative to the fluid along its axis of symmetry. Find the shape of this solid such that the resistance to

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[96] Again, “solid” here means 3D continuous object with no cavity.

[97] For sphere, $\sigma = 4\pi R^2$ and $V = \frac{4}{3}\pi R^3$ (where σ is its surface area, V is its volume, and R is its radius) which can be combined to obtain $\sigma = 4\pi \left(\frac{3V}{4\pi}\right)^{2/3}$ and hence the surface area increases/decreases as the volume increases/decreases (and vice versa).

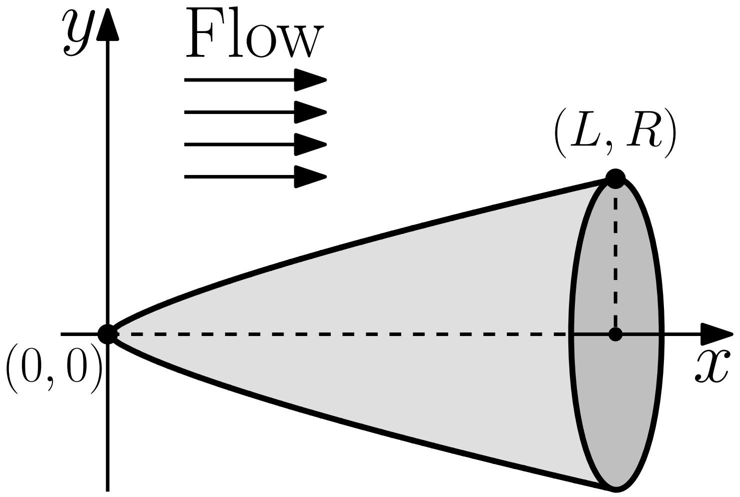

Figure 54: A simple sketch depicting the setting of Problem 1 of § 4.9 where a solid of revolution offers minimum resistance to fluid flow.

the flow is minimum. Use the assumption that the resistance $\mathcal{R}$ experienced by such a solid is given by:

$$\mathcal{R} \propto \int_0^L y y'^3 \, dx$$

where the profile of the object is described by the function $y = y(x)$ with the boundary conditions $y(0) = 0$ and $y(L) = R$ and where the flow is in the positive x direction (see Figure 54).

Answer: From the given information we can write:

$$\mathcal{R} = C \int_0^L y y'^3 \, dx \equiv I[y]$$

where C is a constant, $\mathcal{R}$ (which is the resistance to fluid flow) represents the functional integral I that should be minimized, and y (which is the profile of the solid of revolution and hence it completely determines its shape subject to the boundary conditions) represents the minimizing function that we want to find. Accordingly, $F(x, y, y') = y y'^3$ which is independent of x and hence we can use the Beltrami identity (i.e. Eq. 3), that is:

$$\begin{aligned}
\left(y y'^3\right) - y' \frac{\partial}{\partial y'} \left(y y'^3\right) &= C \\
y y'^3 - y' \left(3 y y'^2\right) &= C \\
y y'^3 - 3 y y'^3 &= C \\
y y'^3 &= C_1 && (C_1 = -C/2) \\
y^{1/3} y' &= C_2 && (C_2 = C_1^{1/3}) \\
y^{1/3} dy &= C_2 dx \\
\frac{3}{4} y^{4/3} &= C_2 x + C_3 \\
y^{4/3} &= C_4 x + C_5 \\
y &= \left(C_4 x + C_5\right)^{3/4}
\end{aligned}$$

Now, from the boundary condition $y(0) = 0$ we have $C_5 = 0$ and from the boundary condition $y(L) = R$ we have $C_4 = R^{4/3}/L$. Hence, the shape of the resistance-minimizing solid of revolution is given by its profile curve:

$$y = \left(\frac{R^{4/3}}{L} x \right)^{3/4} = R \left(\frac{x}{L} \right)^{3/4}$$

2. Re-solve Problem 1 but assume this time that the resistance $\mathcal{R}$ experienced by the solid object is given by:

$$\mathcal{R} \propto \int_0^L y^2 y'^2 \, dx$$

Also, plot the profile of the object assuming that the boundary conditions are $y(0) = 0$ and $y(2) = 1$. **Answer**: Following in our footsteps in Problem 1 we have $F(x, y, y') = y^2 y'^2$ and hence the Euler-Lagrange equation in this case is (see Eq. 3):

$$\begin{aligned}
\left(y^2 y'^2\right) - y' \frac{\partial}{\partial y'} \left(y^2 y'^2\right) &= C \\
y^2 y'^2 - y' \left(2 y^2 y'\right) &= C \\
y^2 y'^2 - 2 y^2 y'^2 &= C \\
y^2 y'^2 &= C_1 \\
y y' &= C_2 \\
y dy &= C_2 dx \\
\frac{1}{2} y^2 &= C_2 x + C_3 \\
y &= \pm \sqrt{C_4 x + C_5}
\end{aligned}$$

Now, from the boundary condition $y(0) = 0$ we have $C_5 = 0$ and from the boundary condition $y(2) = 1$ we have $C_4 = 1/2$. Hence, the profile of the object is given by $y = \pm\sqrt{x/2}$ and it is plotted in Figure 55.

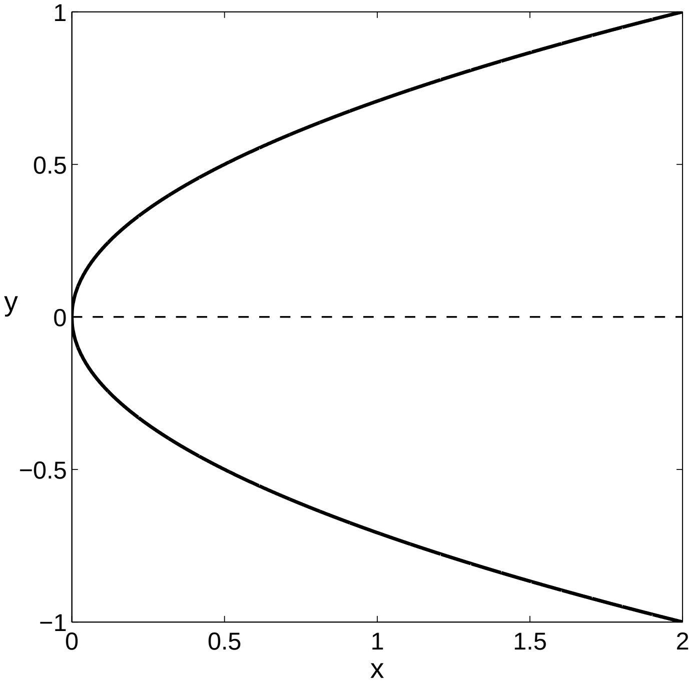

Figure 55: A simple sketch depicting the profile $y = \pm\sqrt{x/2}$ of the solid of revolution of minimum resistance to fluid flow according to Problem 2 of § 4.9.

Chapter 5
Geometrical Optics

One of the main foundations of the classical geometrical optics is Fermat's principle which states that light travels along paths that take least time, and hence this principle is essentially a variational principle. For example, in a vacuum or in a uniform medium the light should propagate along straight paths because least time (according to Fermat's principle) implies least distance (noting that the shortest paths in Euclidean spaces are straight lines and that the speed of light is fixed). Although this principle is not strictly correct[98] and does normally apply only in geometrical optics (where the path of light is much longer than its wavelength), it still has many useful applications.

In this fairly short chapter we investigate a number of variational Problems related to common (and simple) applications of geometrical optics. These Problems are essentially based on Fermat's principle of least time. It is noteworthy that some of these Problems are so simple that they can (and will) be solved by ordinary calculus with no need for the variational formulation of the calculus of variations (as represented by the Euler-Lagrange equation in its various forms and shapes).

Problems

1. Use Fermat's principle to conclude that light in vacuum and homogeneous media follows straight path.
   Answer: This rather trivial result can be easily concluded from Fermat's principle because in Euclidean space (which is the space of classical geometrical optics) the shortest path connecting two points (directly) is straight line (see § 2.1). Now, since in vacuum and homogeneous media the speed of light is constant then least time (according to Fermat's principle) means shortest distance[99] which is the (length of) straight line according to the aforementioned Euclidean geometrical fact.
2. Generalize the result of Problem 1 to include non-Euclidean spaces.
   Answer: The equivalent to "straight line" in Euclidean spaces is "geodesic" in non-Euclidean spaces (or indeed more general). So, by the logic of Problem 1 Fermat's principle should lead to the conclusion that light in vacuum and homogeneous media follows geodesic paths in general (i.e. both in Euclidean spaces and in non-Euclidean spaces).[100]
   Note: it should be noted that in this Problem (as well as in Problem 1) we are assuming direct propagation of light without being affected by subsidiary phenomena (like reflection) that divert the light from its original geodesic trajectory.
3. Derive the law of light reflection (i.e. the angle of incidence is equal to the angle of reflection) using Fermat's principle.
   Answer: In this Problem the light is presumed to propagate in a single uniform medium (or in vacuum) and hence it has a constant speed throughout its journey from the point of departure A to the point of destination C passing through the point of reflection B (see Figure 56).[101] Accordingly, least time (according to Fermat's principle) is equivalent to least distance of travel. So, what we need to do to solve this Problem is to find the path ABC of minimum length by considering the variations

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[98] In fact, this principle may be rectified by changing it from being a minimization principle (as implied by *least* time) to be a principle for obtaining stationary points (where these stationary points can be minimum or maximum or inflection or saddle). The details (which are essentially of physical significance and hence they are of little interest to us as mathematicians of variation) should be sought in the literature.

[99] This is due to the direct relation between distance and time, i.e. $d = ct$ where d is distance, c is speed and t is time.

[100] In fact, we need to provide a rigorous definition for "homogeneous" in non-Euclidean spaces (at least over certain patches of the space). We should also consider possible position dependency of the speed and its local and global significance.

[101] The presumption that the path between A and B and between B and C is straight is based on the result of Problem 1. So, in reality we are applying Fermat's principle twice: once on the sub-paths AB and BC, and once on the entire path ABC. In other words: in this Problem we are using Fermat's principle in both propagation (along AB and BC) and reflection (along ABC).

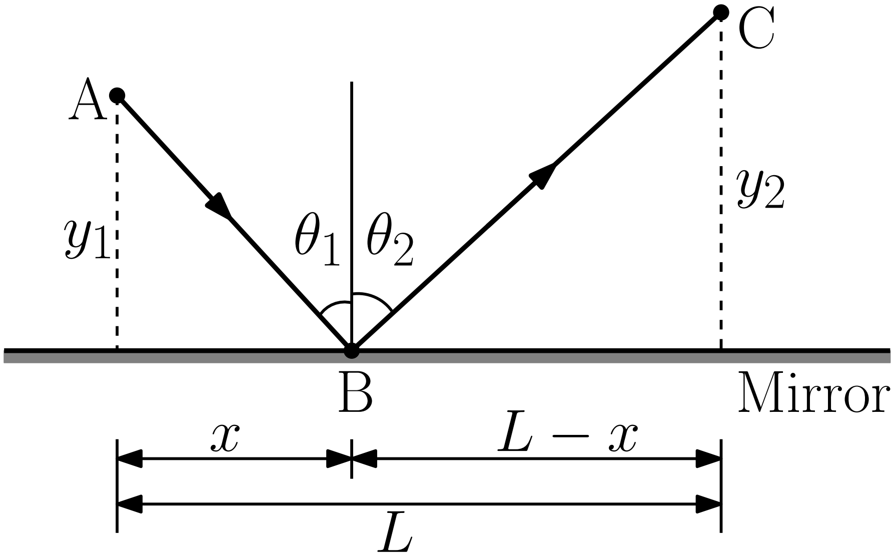

Figure 56: A simple sketch depicting the setting of the derivation of the law of light reflection (see Problem 3 of § 5).

of the angle of incidence θ_1 and the angle of reflection θ_2 (noting that these angles vary as the point of reflection B varies and hence they are functions of the position of B and are correlated to the length of the path ABC). In fact, this Problem can (and will) be solved using simple geometry, algebra and ordinary calculus (supported by the variational principle) with no need for the calculus of variations (as normally represented by the Euler-Lagrange equation).

Now, the length s of the path ABC is the sum of the length of AB and the length of BC where each one of these lengths can be obtained from the Pythagoras theorem, that is:

$$s = \sqrt{x^2 + y_1^2} + \sqrt{(L-x)^2 + y_2^2}$$

So, by the variational principle s should be stationary at its extremum and hence:

$$\begin{aligned}
\frac{ds}{dx} &= 0 \\
\frac{2x}{2\sqrt{x^2+y_1^2}} + \frac{-2(L-x)}{2\sqrt{(L-x)^2+y_2^2}} &= 0 \\
\frac{x}{\sqrt{x^2+y_1^2}} - \frac{(L-x)}{\sqrt{(L-x)^2+y_2^2}} &= 0 \\
\sin\theta_1 - \sin\theta_2 &= 0 \qquad \text{(see Figure 56)} \\
\sin\theta_1 &= \sin\theta_2 \\
\theta_1 &= \theta_2 \qquad \text{(considering the restrictions } 0 < \theta_1, \theta_2 < \frac{\pi}{2}\text{)}
\end{aligned}$$

which is the law of light reflection.

Note 1: it should be obvious that the optimal path in this Problem is a minimum because the path of light (and hence the time of travel) can diverge when point B moves far away.[102] However, in the following we establish this technically by testing the sign of the second derivative of s, that is:

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[102] As we noted earlier in a similar context, although this sort of arguments may not rule out the possibility of a local maximum it is still useful in supporting our intuitive conclusion (noting the secondary importance of issues like this for our variational objectives).

$$\begin{aligned}
\frac{d^2s}{dx^2} &= \frac{1}{\sqrt{x^2+y_1^2}} - \frac{x^2}{\left(x^2+y_1^2\right)^{3/2}} + \frac{1}{\sqrt{(L-x)^2+y_2^2}} - \frac{(L-x)^2}{\left[(L-x)^2+y_2^2\right]^{3/2}} \\
&= \frac{x^2+y_1^2-x^2}{\left(x^2+y_1^2\right)^{3/2}} + \frac{(L-x)^2+y_2^2-(L-x)^2}{\left[(L-x)^2+y_2^2\right]^{3/2}} \\
&= \frac{y_1^2}{\left(x^2+y_1^2\right)^{3/2}} + \frac{y_2^2}{\left[(L-x)^2+y_2^2\right]^{3/2}} \qquad > 0
\end{aligned}$$

So, the optimal s is a minimum and hence the obtained result is consistent with Fermat's principle (even in its restricted form as a principle of *least* time).

Note 2: in the above formulation and arguments we assumed implicitly that the path of light is in a plane that is perpendicular to the plane of the mirror (where the latter plane is the tangent plane if the mirror is not flat). This may also be justified by Fermat's principle (noting that the path in other planes is not optimal relative to the path in the perpendicular plane even if the condition $\theta_1 = \theta_2$ is satisfied).

4. A light ray is emitted from point $(3, 12, 5)$ $\big[$in a 3D Euclidean space coordinated by a Cartesian system$\big]$ toward a flat mirror in the xy plane. If the reflection of this ray is required to pass through the point $(9, 13, 45)$, what the initial direction of the ray should be?
   Answer: If the mirror does not exist then it is required (due to the symmetry and the law of reflection) that the ray should pass through the point $(9, 13, -45)$. Hence, the initial direction should be along the line passing through the initial point $(3, 12, 5)$ and the final point $(9, 13, -45)$, i.e. along the vector $(9-3, 13-12, -45-5) = (6, 1, -50)$.
5. Derive Snell's law of light refraction using Fermat's principle.
   Answer: Snell's law of refraction states that the path of a light ray crossing the boundary between two propagation media of different refractive indices[103] satisfies the following relation (see Figure 57):

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2} = \frac{n_2}{n_1}$$

where θ_1 and θ_2 are the angles of incidence and refraction, c_1 and c_2 are the speeds of light in medium 1 and in medium 2, and n_1 and n_2 are the refractive indices of medium 1 (where the source of light) and medium 2 (where the receiver of light). So, let see how this law can be derived from Fermat's principle.

The length s of the path ABC is the sum of the length s_1 of AB and the length s_2 of BC where each one of these lengths can be obtained from the Pythagoras theorem, that is:[104]

$$s = s_1 + s_2 = \sqrt{x^2+y_1^2} + \sqrt{(L-x)^2+y_2^2}$$

The total time of travel t (which we should minimize according to Fermat's principle) is the sum of the time t_1 along AB and the time t_2 along BC, that is:

$$t = t_1 + t_2 = \frac{s_1}{c_1} + \frac{s_2}{c_2} = \frac{\sqrt{x^2+y_1^2}}{c_1} + \frac{\sqrt{(L-x)^2+y_2^2}}{c_2}$$

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[103] The refractive index n of a medium (for light propagation) is the ratio of the speed of light in vacuum c to the speed of light in that medium c_m, that is $n = c/c_m$. We should note that both media are presumed optically homogeneous; moreover one of the media (and possibly both as a special case) could be vacuum (whose refractive index is 1).

[104] Again, the presumption that the path between A and B and between B and C is straight is based on the result of Problem 1. So, in this Problem we are actually using Fermat's principle in both propagation and refraction.

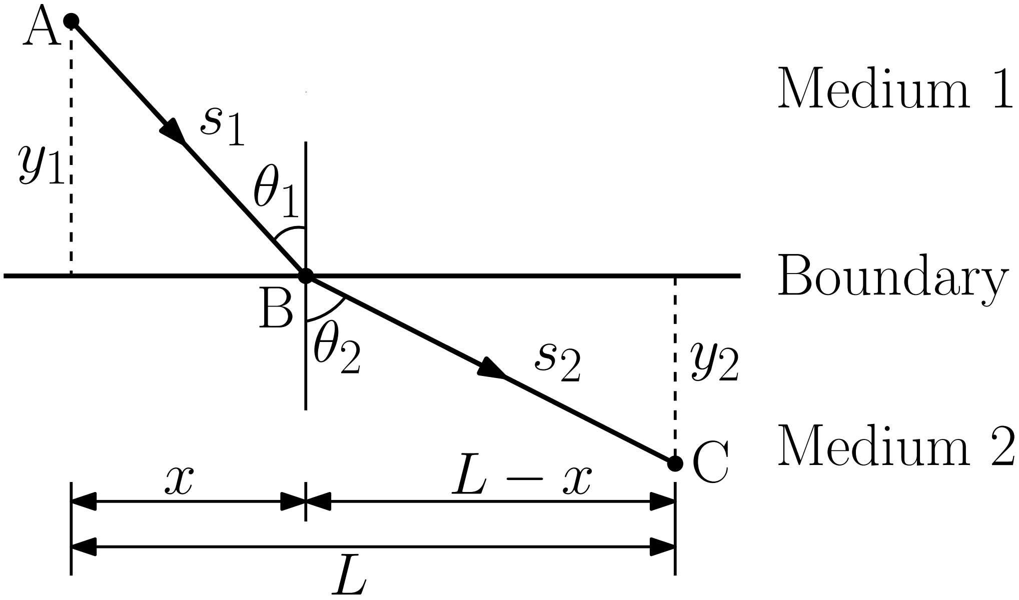

Figure 57: A simple sketch depicting the setting of Problem 5 of § 5 regarding the derivation of Snell's law of light refraction.

Now, by the variational principle t should be stationary at its extremum and hence:

$$\begin{aligned}
\frac{dt}{dx} &= 0 \\
\frac{x}{c_1\sqrt{x^2+y_1^2}} - \frac{(L-x)}{c_2\sqrt{(L-x)^2+y_2^2}} &= 0 \\
\frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2} &= 0 \qquad \text{(see Figure 57)} \\
\frac{\sin\theta_1}{\sin\theta_2} &= \frac{c_1}{c_2} \\
\frac{\sin\theta_1}{\sin\theta_2} &= \frac{c/n_1}{c/n_2} \\
\frac{\sin\theta_1}{\sin\theta_2} &= \frac{n_2}{n_1} \qquad (111)
\end{aligned}$$

which is Snell's law of light refraction.

Note: again, the optimal path in this Problem is obviously a minimum because the path of light (and hence the time of travel) can diverge when point B moves far away. This can also be established technically by testing the sign of the second derivative of t (as done for s in Problem 3).

6. A light ray is emitted from point $(5,3)$ $\left[\text{in a plane coordinated by a Cartesian system}\right]$ and it is required to reach point $(10,-23)$. If the refractive index for $y>0$ is $n_1 = 1.2$ and the refractive index for $y<0$ is $n_2 = 1.35$, what are the coordinates of the point on the boundary through which the ray passes? Also, what are the angle of incidence θ_1 and the angle of refraction θ_2?

   Answer: Referring to Problem 5 (and Figure 57), we have $L = 10-5 = 5$, $\sin\theta_1 = \frac{x}{\sqrt{x^2+3^2}}$ and $\sin\theta_2 = \frac{5-x}{\sqrt{(5-x)^2+(-23)^2}}$. Hence, from Eq. 111 we get:

$$\frac{x}{\sqrt{x^2+3^2}} \times \frac{\sqrt{(5-x)^2+(-23)^2}}{5-x} = 1.125$$

$$x \simeq 0.6424877425$$

So, the coordinates are $(5.6424877425, 0)$. Also:

$$\begin{aligned}
\theta_1 &= \arcsin\left(\frac{0.6424877425}{\sqrt{0.6424877425^2+3^2}}\right) \simeq 0.210976\,\text{rad} \simeq 12.088012^\circ \\
\theta_2 &= \arcsin\left(\frac{5-0.6424877425}{\sqrt{(5-0.66424877425)^2+(-23)^2}}\right) \simeq 0.187271\,\text{rad} \simeq 10.729814^\circ
\end{aligned}$$

7. Find the shape of the path traced by a ray of light moving in a material medium whose refractive index is proportional to y ($y > 0$; see Figure 58).[105]

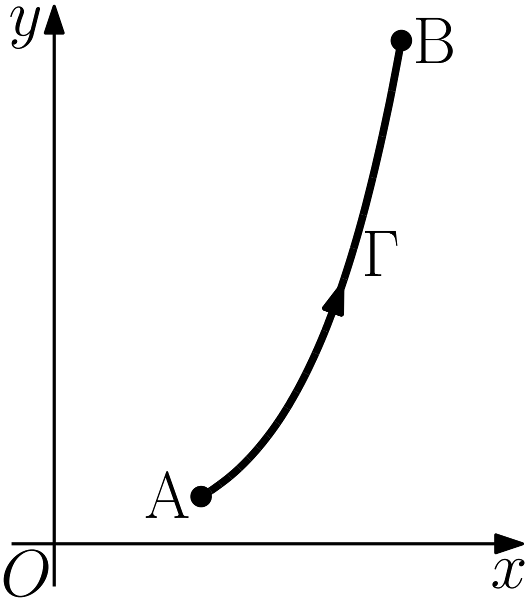

Figure 58: A schematic illustration of the setting of Problem 7 of § 5 where a light ray moves from point A(x_A, y_A) to point B(x_B, y_B) along the path Γ (which is confined to the xy plane) in a material medium whose refractive index is proportional to y (i.e. $n = ay$ with a being a positive constant).

Answer: According to Fermat's principle the time t for the propagation of the light ray between point A (source) and point B (destination) should be a minimum. So, our minimized functional $I[y]$ should be an integral representing t, that is:

$$t = \int_\Gamma dt = \int_\Gamma \frac{ds}{v} = \int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}dx}{c_m} = \int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}dx}{c/(ay)} = \frac{a}{c}\int_{x_A}^{x_B} y\sqrt{1+y'^2}\,dx \equiv I[y]$$

where in step 2 we used $v = ds/dt$ (and hence $dt = ds/v$), in step 3 we used $ds = \sqrt{1+y'^2}dx$ and $v \equiv c_m$ (with c_m being the speed of light in the material medium), and in step 4 we used the fact that the speed of light c_m in a material medium with refractive index n is given by c/n $\big[$noting that in this case $n = ay$ (with a being a positive constant) because the refractive index is proportional to

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[105] In this Problem (and other similar Problems which will follow) we are assuming that the speed of light (and hence the refractive index) is a function of position but not of direction, i.e. the medium is optically isotropic although it is not homogeneous. We also consider the path of the ray to be confined to the xy plane. It should also be noted that the condition $y > 0$ does not impose any real restriction on the formulation of this Problem (neither physical nor mathematical) because we can always choose our coordinate system so that the entire path is in the region $y > 0$ (although $y < 0$ or using a negative proportionality constant may be dealt with as a different problem).

$y]$. As we see, F in this Problem is identical to F in the problem of catenary (see Problem 1 of § 2.3). Accordingly, the shape of the light path should also be a hyperbolic cosine, that is:

$$y = C\cosh\left(\frac{x-D}{C}\right) \tag{112}$$

8. Find the shape of the path traced by a ray of light moving in a material medium whose refractive index is proportional to $1/y$ ($y > 0$; see Figure 59).

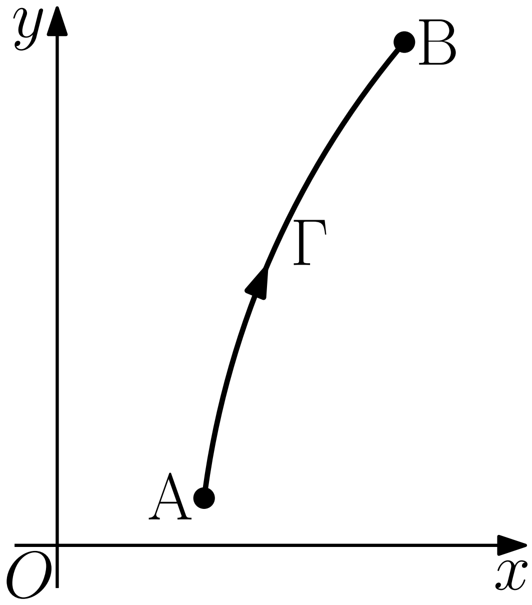

Figure 59: A schematic illustration of the setting of Problem 8 of § 5 where a light ray moves from point $\text{A}(x_A, y_A)$ to point $\text{B}(x_B, y_B)$ along the path Γ (which is confined to the xy plane) in a material medium whose refractive index is proportional to $1/y$ (i.e. $n = b/y$ with b being a positive constant).

Answer: On following the reasoning and method of Problem 7 we get:

$$t = \int_\Gamma dt = \int_\Gamma \frac{ds}{v} = \int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}dx}{c_m} = \int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}dx}{c/(b/y)} = \frac{b}{c}\int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}}{y}\,dx \equiv I[y]$$

where b is a positive constant. As we see, $F = \frac{\sqrt{1+y'^2}}{y}$ and hence the Euler-Lagrange equation (see Eq. 3) is:

$$\begin{aligned}
\frac{\sqrt{1+y'^2}}{y} - y'\frac{\partial}{\partial y'}\left(\frac{\sqrt{1+y'^2}}{y}\right) &= C \\
\frac{\sqrt{1+y'^2}}{y} - y'\left(\frac{y'}{y\sqrt{1+y'^2}}\right) &= C \\
\frac{\sqrt{1+y'^2}}{y} - \frac{y'^2}{y\sqrt{1+y'^2}} &= C \\
\frac{1+y'^2}{y\sqrt{1+y'^2}} - \frac{y'^2}{y\sqrt{1+y'^2}} &= C \\
\frac{1}{y\sqrt{1+y'^2}} &= C
\end{aligned}$$

$$\begin{aligned}
y^2\left(1+y'^2\right) &= \frac{1}{C^2} \\
y'^2 &= \frac{1-C^2y^2}{C^2y^2} \\
y' &= \pm\sqrt{\frac{1-C^2y^2}{C^2y^2}} \\
\pm\sqrt{\frac{C^2y^2}{1-C^2y^2}}dy &= dx \\
\mp\frac{1}{C}\sqrt{1-C^2y^2} &= x+D \\
\frac{1}{C^2}\left(1-C^2y^2\right) &= (x+D)^2 \\
(x+D)^2+y^2 &= \frac{1}{C^2} \qquad (113)
\end{aligned}$$

So, the shape of the light path is a circular arc centered on the x axis.

9. Find the shape of the path traced by a ray of light moving in a material medium of refractive index $n = e^y$ $(y > 0)$.
   Answer: Following in our footsteps in the previous Problems we get:

$$t = \int_\Gamma dt = \int_\Gamma \frac{ds}{v} = \int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}dx}{c_m} = \int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}dx}{c/e^y} = \frac{1}{c}\int_{x_A}^{x_B} e^y\sqrt{1+y'^2}\,dx \equiv I[y]$$

As we see, $F = e^y\sqrt{1+y'^2}$ and hence the Euler-Lagrange equation (see Eq. 3) is:

$$\begin{aligned}
e^y\sqrt{1+y'^2} - y'\frac{\partial}{\partial y'}\left(e^y\sqrt{1+y'^2}\right) &= C \\
e^y\sqrt{1+y'^2} - \frac{e^y y'^2}{\sqrt{1+y'^2}} &= C \\
\frac{e^y\left(1+y'^2\right)}{\sqrt{1+y'^2}} - \frac{e^y y'^2}{\sqrt{1+y'^2}} &= C \\
\frac{e^y}{\sqrt{1+y'^2}} &= C \\
e^{2y} &= C^2\left(1+y'^2\right) \\
y'^2 &= \frac{e^{2y}-C^2}{C^2} \\
y' &= \frac{\sqrt{e^{2y}-C^2}}{C} \\
\frac{C\,dy}{\sqrt{e^{2y}-C^2}} &= dx \\
\arctan\left(\frac{\sqrt{e^{2y}-C^2}}{C}\right) &= x+D \qquad (114)
\end{aligned}$$

10. Find the shape of the path traced by a ray of light moving in a material medium of refractive index $n = a/\rho$ where a is a positive constant and ρ is a polar coordinate (assuming the path being in a plane coordinated by polar coordinates ρ, ϕ).
    Answer: Following in our footsteps in the previous Problems (but using polar coordinates) we get:

$$t \;=\; \int_\Gamma dt = \int_\Gamma \frac{ds}{v} = \int_\Gamma \frac{\sqrt{(d\rho)^2+\rho^2(d\phi)^2}}{c_m} = \int_{\rho_A}^{\rho_B} \frac{\sqrt{1+\rho^2(d\phi/d\rho)^2}\,d\rho}{c/(a/\rho)}$$

$$= \frac{a}{c}\int_{\rho_A}^{\rho_B} \frac{\sqrt{1+\rho^2\phi'^2}}{\rho}\,d\rho \equiv I[\phi]$$

where in step 3 we used the expression for the line element ds in polar coordinates and $v \equiv c_m$. As we see, $F(\rho,\phi,\phi') = \rho^{-1}\sqrt{1+\rho^2\phi'^2}$ and hence the Euler-Lagrange equation (see Eq. 4 noting the correspondence between x, y, y' and ρ, ϕ, ϕ') is:

$$\begin{aligned}
\frac{\partial}{\partial\phi'}\left(\rho^{-1}\sqrt{1+\rho^2\phi'^2}\right) &= C \\
\rho^{-1}\frac{\rho^2\phi'}{\sqrt{1+\rho^2\phi'^2}} &= C \\
\frac{\rho\phi'}{\sqrt{1+\rho^2\phi'^2}} &= C \\
\rho^2\phi'^2 &= C^2\left(1+\rho^2\phi'^2\right) \\
\phi'^2 &= \frac{C^2}{\rho^2 - C^2\rho^2} \\
\phi' &= \frac{D}{\rho} \qquad \left(D = \pm\sqrt{C^2/[1-C^2]}\right) \\
\phi &= D\ln\rho + E \qquad (115)
\end{aligned}$$

11. Find the shape of the path traced by a ray of light moving in a material medium of refractive index $n = \frac{a\rho^4\phi'^3}{\sqrt{1+\rho^2\phi'^2}}$ where a is a positive constant, ρ is a polar coordinate (assuming the path being in a plane coordinated by polar coordinates ρ, ϕ) and $\phi' = d\phi/d\rho$.
Answer: As in the previous Problem, we have:

$$t = \int_\Gamma dt = \int_\Gamma \frac{ds}{v} = \int_\Gamma \frac{\sqrt{(d\rho)^2 + \rho^2(d\phi)^2}}{c_m} = \int_{\rho_A}^{\rho_B} \frac{\sqrt{1+\rho^2\phi'^2}\,d\rho}{c\left(\frac{\sqrt{1+\rho^2\phi'^2}}{a\rho^4\phi'^3}\right)} = \frac{a}{c}\int_{\rho_A}^{\rho_B} \rho^4\phi'^3\,d\rho \equiv I[\phi]$$

As we see, $F(\rho,\phi,\phi') = \rho^4\phi'^3$ and hence the Euler-Lagrange equation (see Eq. 4 noting the correspondence between x, y, y' and ρ, ϕ, ϕ') is:

$$\begin{aligned}
\frac{\partial}{\partial\phi'}\left(\rho^4\phi'^3\right) &= C \qquad (C > 0) \\
3\rho^4\phi'^2 &= C \\
\phi'^2 &= \frac{C}{3\rho^4} \\
\phi' &= \frac{\pm\sqrt{C/3}}{\rho^2} \\
\phi &= \frac{D}{\rho} + E \qquad \left(D = \mp\sqrt{C/3}\right) \qquad (116)
\end{aligned}$$

12. A light ray is emitted from point $(7, 11)$ [in a plane coordinated by a Cartesian system] and it is required to reach point $(8, 37)$. If the refractive index of the medium of propagation is proportional to y, what the initial direction of the ray should be? Also, plot the trajectory between the two points.
Answer: According to the result of Problem 7, the path is given by Eq. 112 and hence the points $(7, 11)$ and $(8, 37)$ should satisfy this equation. On solving Eq. 112 for x and substituting the two points in the resulting equation we get:

$$7 = C\,\text{arccosh}\left(\frac{11}{C}\right) + D$$

$$8 = C\,\text{arccosh}\left(\frac{37}{C}\right) + D$$

On subtracting the first of these equations from the second we get $1 = C\,\text{arccosh}\left(\frac{37}{C}\right) - C\,\text{arccosh}\left(\frac{11}{C}\right)$ which can be solved numerically to obtain $C \simeq 0.823517739$. On inserting this value of C into one of the above equations we get $D \simeq 4.295725460$. So, the trajectory of the light ray is given by:

$$y = 0.823517739\cosh\left(\frac{x - 4.295725460}{0.823517739}\right)$$

Accordingly, the initial direction can be determined from the slope of the trajectory at point $(7, 11)$, that is:

$$\left.\frac{dy}{dx}\right|_{x=7} = \sinh\left(\frac{x - 4.295725460}{0.823517739}\right)\Bigg|_{x=7} \simeq 13.319846963$$

The trajectory is plotted in Figure 60.

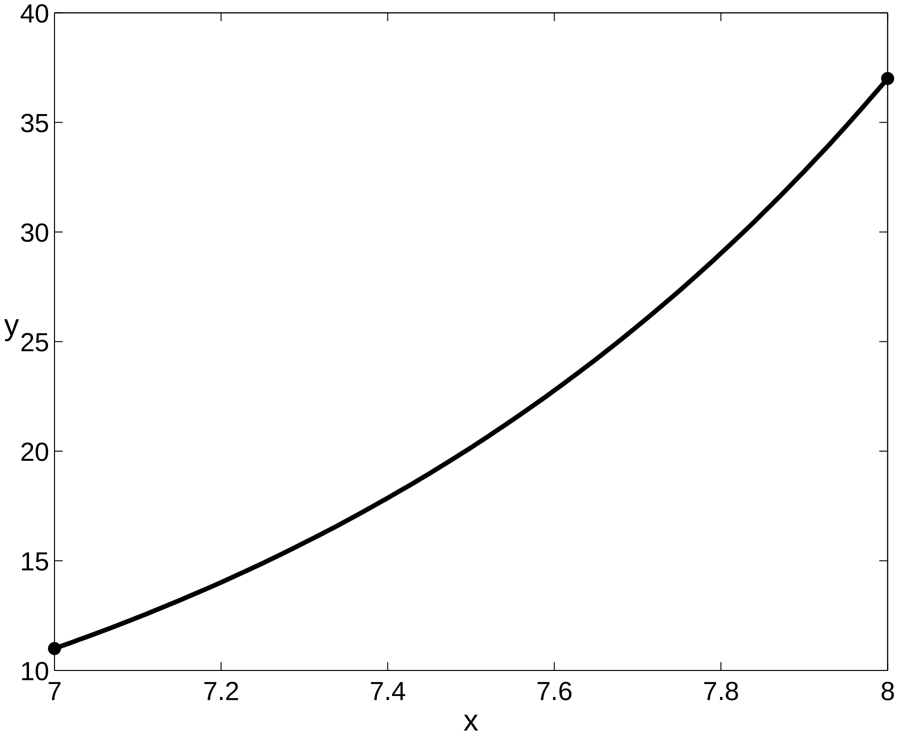

Figure 60: Plot of the trajectory $y = 0.823517739\cosh\left(\frac{x-4.295725460}{0.823517739}\right)$ of the light ray of Problem 12 of § 5.

13. A light ray is emitted from point $(7, 11)$ [in a plane coordinated by a Cartesian system] and it is required to reach point $(8, 37)$. If the refractive index of the medium of propagation is proportional to $1/y$, find the equation of the trajectory of the ray.
Answer: According to the result of Problem 8, the trajectory is given by Eq. 113 and hence the points $(7, 11)$ and $(8, 37)$ should satisfy this equation, that is:

$$\begin{aligned}(7 + D)^2 + 11^2 &= \frac{1}{C^2} \\ (8 + D)^2 + 37^2 &= \frac{1}{C^2}\end{aligned}$$

On subtracting the first of these equations from the second we get $(8+D)^2+37^2-(7+D)^2-11^2=0$ which can be solved to obtain $D=-1263/2$. On inserting this value of D into one of the above equations we get $C=\sqrt{4/1560485}$. So, the equation of the trajectory is:

$$\left(x-\frac{1263}{2}\right)^2+y^2=\frac{1560485}{4}$$

which is a circular arc (as concluded in Problem 8).

14. A light ray is emitted from point $(0,1)$ $\big[$in a plane coordinated by a Cartesian system$\big]$ in the direction $(1,2)$. If the refractive index of the medium of propagation is $n=e^y$, determine if the ray will pass through the point $(0.3,2)$. Also, plot the trajectory for the range $y=1$ to $y=5$.
Answer: According to the result of Problem 9, the slop of the trajectory is $y'=\frac{\sqrt{e^{2y}-C^2}}{C}$. So, at point $(0,1)$ the slope is:

$$\begin{aligned} y'\Big|_{y=1} &= \frac{\sqrt{e^2-C^2}}{C} \\ \frac{2}{1} &= \frac{\sqrt{e^2-C^2}}{C} \\ 4C^2 &= e^2-C^2 \\ C &= \frac{e}{\sqrt{5}} \end{aligned}$$

So, from Eq. 114 the trajectory is given by:

$$\begin{aligned} \arctan\left(\frac{\sqrt{e^{2y}-\left(e/\sqrt{5}\right)^2}}{e/\sqrt{5}}\right) &= x+D \\ \arctan\left(\sqrt{5e^{2(y-1)}-1}\right) &= x+D \end{aligned}$$

Now, since the ray is emitted from point $(0,1)$ this point should satisfy this equation, that is:

$$\begin{aligned} \arctan\left(\sqrt{5e^{2(1-1)}-1}\right) &= 0+D \\ D &= \arctan(2) \end{aligned}$$

Accordingly, the trajectory is given (implicitly) by:

$$\begin{aligned} \arctan\left(\sqrt{5e^{2(y-1)}-1}\right) &= x+\arctan(2) \\ x &= \arctan\left(\sqrt{5e^{2(y-1)}-1}\right)-\arctan(2) \end{aligned}$$

The point $(0.3,2)$ does not satisfy this equation and hence it is not on the trajectory (but it is very close). The trajectory for the range $y=1$ to $y=5$ is plotted in Figure 61.

15. Make a polar plot for the trajectory of the light ray in Problem 11 for the range $\phi=\pi/3$ to $\phi=\pi$ assuming the ray passes through the points with polar coordinates $(3,\pi/3)$ and $(1,\pi)$.
Answer: Inserting the coordinates of the given points in Eq. 116 we get:

$$\begin{aligned} \frac{\pi}{3} &= \frac{D}{3}+E \\ \pi &= D+E \end{aligned}$$

On subtracting the first equation from the second we get $\frac{2\pi}{3}=\frac{2D}{3}$ and hence $D=\pi$. On inserting this value of D into one of the above equations we get $E=0$. Hence, the equation of the trajectory is:

$$\rho=\frac{\pi}{\phi} \qquad \left(\frac{\pi}{3}\le\phi\le\pi\right)$$

The plot is shown in Figure 62.

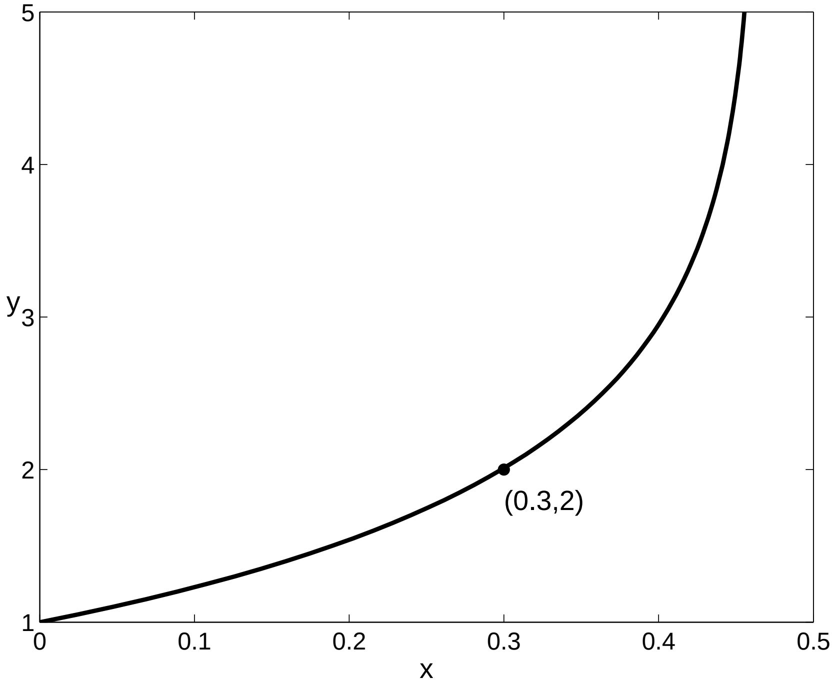

Figure 61: Plot of the trajectory $x = \arctan\left(\sqrt{5e^{2(y-1)} - 1}\right) - \arctan(2)$ of the light ray of Problem 14 of § 5. The point $(0.3, 2)$ which is not on the trajectory (but it is very close) is marked.

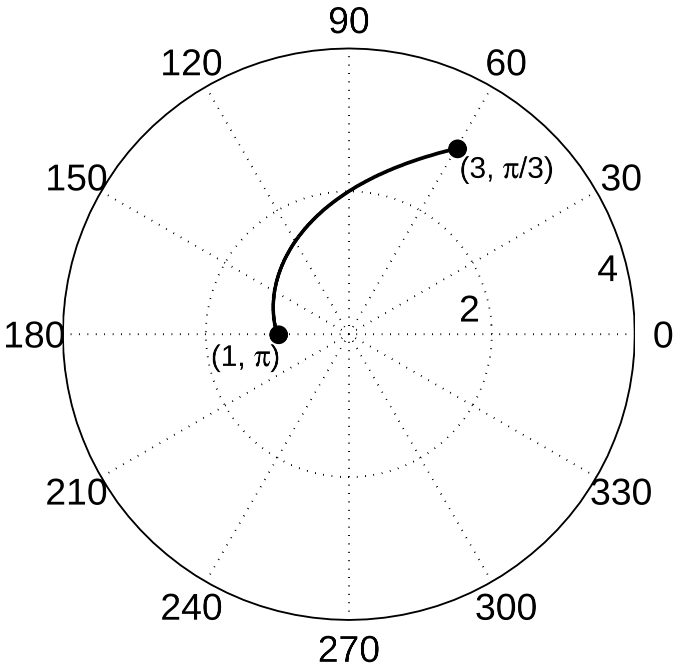

Figure 62: The polar plot of the trajectory $\rho = \frac{\pi}{\phi}$ of the light ray of Problem 15 of § 5 for the range $\phi = \pi/3$ to $\phi = \pi$. The numbers on the perimeter are the polar angle ϕ in degrees while the numbers 2 and 4 are the $\rho = 2$ and $\rho = 4$ circles.

Chapter 6
Hamiltonian Mechanics

One of the biggest fields of application (as well as development) of the mathematics of variation (and the calculus of variations in particular) is mechanics where this subject in its variational form was historically developed by William Hamilton and hence it is commonly known as the Hamiltonian mechanics.[106] In the Hamiltonian mechanics distinctive terminology is used. For example, the functional integral I in this mechanics is called the "action" and hence the principle of minimizing this integral (which essentially reflects the spirit of the variational principle) is called "Hamilton's principle of least action".[107] Distinctive notation is also used in the formulation of the variational problems. For example, in the variational problems with n extremizing functions (see § 1.7) the independent variable is usually denoted with t (which normally stands for time) and the extremizing functions are denoted with $q_1(t), \cdots, q_n(t)$ while overdot is used to symbolize the derivatives of these variables with respect to t.[108] Also, the integrand F in the Hamiltonian mechanics is usually denoted with L and is called the Lagrangian (or Lagrangian function). Hence, the action integral is given as:

$$I\left[q_1, \cdots, q_n\right] = \int_{t_1}^{t_2} L\left(t, q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n\right)\, dt \tag{117}$$

Accordingly, a set of n Euler-Lagrange equations is required where these equations are given compactly by:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad\qquad (i = 1, \cdots, n) \tag{118}$$

This set of n second order differential equations (which are commonly known as Lagrange's equations or Lagrangian equations) should be solved to obtain the solution of the variational problem (i.e. $q_1, \cdots, q_n$) where the initial conditions can be used to determine the $2n$ constants of integration that are involved in the solution (noting that in some cases the Hamiltonian formulation can lead to first integrals). It is worth noting that Eq. 117 may be written in a compact form (using vector notation) as:

$$I\left[\mathbf{q}\right] = \int_{t_1}^{t_2} L\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\, dt \tag{119}$$

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[106] It may also be called the Lagrangian mechanics. In fact, Hamilton is not the only or the first mathematician to work on this topic but we attributed the development to him because in this book we are interested in his version of the variational formulation of this field.

[107] Some authors suggest that Hamilton's principle is closely related to the principle of least action but it is not the same. More specifically, Hamilton's principle is seen as more general than the principle of least action. Also, being minimizing (as suggested by "least") is not guaranteed by this principle and hence it may be more appropriate to call it the principle of stationary action (as will be indicated later). In fact, there are many details about these issues and their alike but they are beyond the scope of this book.

[108] We should note that the extremizing functions q_i in the Hamiltonian mechanics usually represent the spatial coordinates (and hence their time derivatives $\dot{q}_i$ represent the velocity components) of a mechanical system with n degrees of freedom where the configuration of the system is described by these n functions and their derivatives. So, the purpose of the variational formulation is to describe the motion of the system and determine its configuration (as a function of time) as a result of the external and internal forces which the system is subject to. In fact, the configuration of the system can be seen as a t-parameterized "curve" in an n dimensional space. Classical systems are deterministic and hence by knowing the configuration at an initial time (by knowing the positions and velocities) it can be determined at any time in the future (in fact this may be generalized by saying: by knowing the configuration at a given time it can be determined at any other time). So, the variational formulation of a given mechanical problem plus the initial conditions should be able to solve the problem completely.

where $\mathbf{q} = (q_1, \cdots, q_n)$ and $\dot{\mathbf{q}} = (\dot{q}_1, \cdots, \dot{q}_n)$. This notation is particularly useful when the above discrete system formulation (with a finite number of degrees of freedom n) is generalized to the continuous system formulation (with an infinite n).

The Hamiltonian mechanics is based on Newton's laws (or on the Newtonian mechanics to be more comprehensive) in its physical framework while it employs the calculus of variations as its main mathematical technique. The fundamental principle of this mechanics is the aforementioned "Hamilton's principle of least action" which states: in a mechanical system subject to conservative forces only the behavior of the system (according to the Newtonian mechanics) is described by an extremal[109] of the action integral $I = \int L\,dt$ where the Lagrangian is given by $L = T - U$ (with T and U being the kinetic and potential energy of the system respectively).[110]

We should finally draw the attention to the following important remarks:

- There are two important generalizations to the Hamiltonian formulation. First, q_i's are not necessarily required to be representing conventional coordinates (i.e. they can be used more generally to represent the variables and physical conditions of the mechanical system and hence they may be called generalized coordinates). Second, as indicated earlier the Hamiltonian formulation is not necessarily required to be for discrete systems (e.g. systems of separate particles) and therefore the Hamiltonian mechanics can be used to formulate even problems of continuous systems such as fluid dynamics and other continuum mechanics systems (see Problem 19).
- Since the Hamiltonian mechanics is a variational subject that is based on the Euler-Lagrange equation (as seen above) it is subject to the variations of the Euler-Lagrange equation (as investigated in § 1.4-1.10) and hence the above formulation of the Hamiltonian mechanics (as represented by Eqs. 117 and 118) represents the common cases. For example, the Hamiltonian formulation may have only one dependent variable (see for instance part a of Problem 4; also see Problems 12 and 19) or it may have multiple independent variables (see Problem 19). In brief, the Hamiltonian mechanics is just an application and instantiation of the calculus of variations as represented by the Euler-Lagrange equation in its various forms and flavors (where the principle of least action plays the major role in establishing and justifying the physics).[111]
- Our objective in the Problems of this chapter is to clarify the variational aspects of the Hamiltonian mechanics (as an application of the calculus of variations), and hence any other issue (such as the physics behind the individual Problems) is not of major concern or interest to us. Accordingly, the presentation and investigation of those other issues may be superficial or non-rigorous.

Problems

1. Discuss the following statement: "The Hamiltonian mechanics is based on Newton's laws in its physical framework".

   Answer: In general terms, the Newton's laws formulation of mechanics and the Hamiltonian formulation of mechanics are independent but equivalent formulations and hence they may be seen as equally fundamental (noting that each one of these formulations can be derived from the other). Yes, Newton's laws have historical precedence and hence the Hamiltonian mechanics can be seen from this perspective as originating from the Newtonian mechanics although the Hamiltonian mechanics may also be seen as more fundamental from other perspectives (as discussed in the literature and will be touched on later). In fact, there are some differences in opinion about which is more fundamental (assuming a precedence in some sense is presumed or established). Anyway, our opinion is that Newton's laws are more fundamental from a theoretical and conceptual perspective due to the fact that the Newtonian philosophy and paradigms (which Newton's laws and Newtonian mechanics are based on) are at the foundation of the Hamiltonian rationale and formulation. On the other hand, the Hamiltonian mechanics can be seen as more fundamental (at least in some cases and situations) from a practical and procedural perspective. It may also be regarded as being more fundamental from its variational perspective since the principle of variation or optimization is one of the most fundamental physical principles due to the fact that many applications and branches of science are (or can be) established from the logic and rationale of optimization and variation. Also, see Problem 2.

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[109] We note that in most cases it is minimal but this is not necessarily the case. In fact, to be more accurate and general we should use "stationary" instead of "extremal" (but we followed what is common in the literature in stating this principle).

[110] We should note that the given statement expresses the essence of this principle only (ignoring some details and conditions which are out of scope) and hence it is not sufficiently rigorous. We should also note that from this statement we can see that the action has the physical dimensions of energy times time.

[111] In other words, the Euler-Lagrange equation provides the mathematics of this mechanics while Hamilton's principle of least action provides the physics of this mechanics.

2. Make a brief comparison between Lagrange's equations in mechanics (as embedded in the above-described formulation of the Hamiltonian mechanics) and Newton's laws of motion.
Answer: We note the following:
• Lagrange's equations are physically and logically equivalent to Newton's laws and hence they can be seen as another form of Newton's laws.
• Lagrange's equations are scalar equations while Newton's laws are essentially vector equations. This may be seen as an advantage to Lagrange's equations since dealing with scalar formulations is generally easier than dealing with vector formulations.
• The difference in the previous point between the two formulations may lead to another difference (and advantage to Lagrange's equations over Newton's laws) related to the invariance of the two formulations where it is claimed that Newton's laws in component form are not manifestly invariant across coordinate systems while Lagrange's equations are invariant.
• Being based on the paradigms of energy and variation (which are general paradigms that occur across various disciplines of physics), the formulation of Lagrange's equations can be easily and naturally extended to fields of physics other than mechanics (and classical mechanics in particular which is the birthplace of this formulation). This may be seen as another advantage to this formulation in comparison to Newton's laws which are less general in application and usefulness. In fact, this advantage can lead to other advantages such as generalization and unification.
• Being essentially a variational formulation based on the notion of action, the formulation of Lagrange's equations is more fit and natural for investigating conservation principles and symmetries in physical systems and the relations between the two. This should be seen as another important advantage to this formulation over the Newtonian formulation.
• Noting that the principle of least (or stationary) action is restricted to conservative systems, Newton's laws of motion may be seen as more general from this perspective. In fact, there are more limitations and restrictions on the Hamiltonian mechanics (and hence on Lagrange's equations of mechanics).[112]
• Newton's laws of motion are associated with (and based on) a certain philosophical and epistemological framework (unlike the formulation of Lagrange's equations which is more like a physical theory or a mathematical method of purely technical nature) and hence Newton's laws have more significance and far-reaching consequences from this theoretical perspective (although this may be questioned).

3. State (briefly and non-rigorously) the principle of least action of the Hamiltonian mechanics.
Answer: In conservative mechanical systems, the particles (possibly within a continuum) follow trajectories that optimize the action integral $I = \int L\,dt$ where L is the Lagrangian defined as the difference between the kinetic and potential energies, i.e. $L = T - U$.

4. Find the Lagrangian of the following mechanical systems:
(**a**) A yo-yo hanging from a ceiling and it is unwinding vertically downward (see Figure 63).
(**b**) A system made of a particle of mass m_1 connected to another particle of mass m_2 by a flexible inextensible string of negligible mass where m_1 is moving on a horizontal table while m_2 is dangling vertically with the string being passed through a tiny hole in the table (see Figure 64). Assume that gravity is the only acting force with no friction involved (neither between m_1 and the table nor between the string and the table or the hole).
Answer:
(**a**) Assume that the mass of the yo-yo string is negligible and the mass of the yo-yo (i.e. its rotating part) is m and its moment of inertia is I. Let the plane of the ceiling represent the zero potential energy reference (see Figure 63). Now, if l is the length of the unwinded part of the yo-yo string then the potential energy of the system is $U = -mgl$. Regarding the kinetic energy, it consists of a

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[112] In fact, generalizations and extensions to the principle of least action and its application should lift some of these limitations and restrictions.

translational part:

$$T_t = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dl}{dt}\right)^2 = \frac{1}{2}m\dot{l}^2$$

and a rotational part:

$$T_r = \frac{1}{2}I\omega^2 = \frac{1}{2}I\left(\frac{v}{R}\right)^2 = \frac{1}{2}I\frac{\dot{l}^2}{R^2}$$

where v is the translational speed of the yo-yo (i.e. its descending part), ω is its angular speed, and R is the radius of its axle (assuming the thickness of the winded part of the string is negligible).[113] Accordingly, the Lagrangian of this mechanical system is:

$$L\left(t, l, \dot{l}\right) = T - U = T_t + T_r - U = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}I\frac{\dot{l}^2}{R^2} + mgl = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{l}^2 + mgl$$

Note: unlike the Lagrangian of most Hamiltonian mechanics problems, the Lagrangian of this problem has only one dependent variable (i.e. l) thanks to the correlation between l and ω which we exploited to simplify the formulation (as well as being essentially a 1D problem).

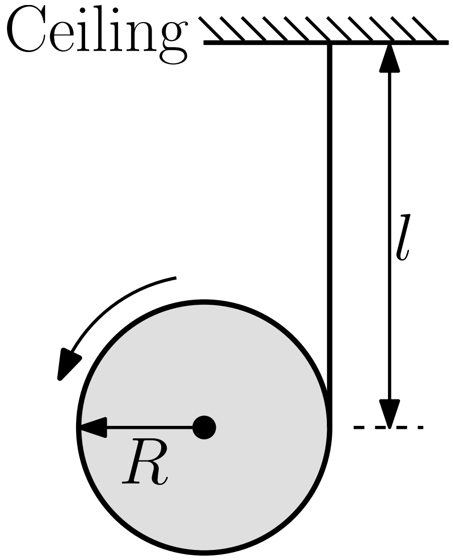

Figure 63: A schematic illustration of the yo-yo mechanical system. See part (a) of Problem 4 of § 6.

(**b**) Let use a cylindrical coordinate system whose origin is at the hole, and its z axis is pointing vertically downward (see Figure 64). Also, let the length of the string be l and the zero potential energy reference be the table level. Accordingly, the coordinates of m_1 are $(\rho, \phi, 0)$ and the coordinates of m_2 are $(0, 0, z)$. However, because the string is inextensible l is constant and hence $\rho + z = l$ which leads to $z = l - \rho$. Now, the potential energy of the system is the sum of the potential energies of its parts. However, because m_1 remains at the zero potential energy level its potential energy is zero and hence the potential energy of the system is the potential energy of m_2 which is $U = -m_2 g z = -m_2 g (l - \rho)$. Regarding the kinetic energy of the system, it is the sum of the kinetic energies of its parts, that is:

$$\begin{aligned} T &= T_1 + T_2 \\ &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \end{aligned}$$

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[113] If the thickness is not negligible then it should be included in R and this leads to more complicated formulation.

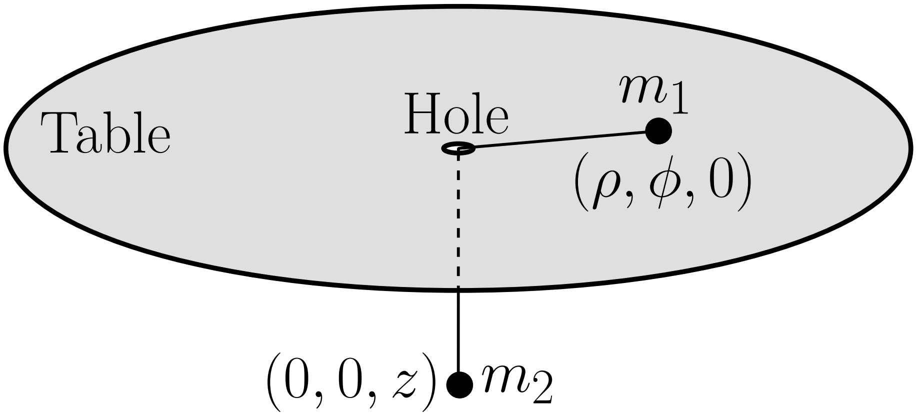

Figure 64: A schematic illustration of the setting of part (b) of Problem 4 of § 6.

$$\begin{aligned}
&= \frac{1}{2}m_1\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + \frac{1}{2}m_2\left(\frac{dz}{dt}\right)^2 && (\mathbf{v}_1 \text{ has radial and azimuthal components}) \\
&= \frac{1}{2}m_1\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + \frac{1}{2}m_2\dot{\rho}^2 && (z = l - \rho)
\end{aligned}$$

Accordingly, the Lagrangian of this mechanical system is:

$$L\left(t, \rho, \phi, \dot{\rho}, \dot{\phi}\right) = T - U = \frac{1}{2}m_1\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + \frac{1}{2}m_2\dot{\rho}^2 + m_2 g\left(l - \rho\right)$$

5. Given that the kinetic energy T and the potential energy U of a mechanical system are given by:

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\dot{q}_i^2 \qquad \text{and} \qquad U = U\left(q_1, \cdots, q_n\right)$$

show that $T + U =$ constant. What this means?
Answer: We have:

$$\begin{aligned}
-\left(T + U\right) &= -T - U \\
&= T - U - 2T \\
&= L - 2T \\
&= L - \sum_{i=1}^{n} m_i\dot{q}_i^2 \\
&= L - \sum_{i=1}^{n} \dot{q}_i\left[m_i\dot{q}_i\right] \\
&= L - \sum_{i=1}^{n} \dot{q}_i\frac{\partial}{\partial\dot{q}_i}\left[\frac{1}{2}\sum_{j=1}^{n} m_j\dot{q}_j^2 - U\right] \\
&= L - \sum_{i=1}^{n} \dot{q}_i\frac{\partial L}{\partial\dot{q}_i} \\
&= C
\end{aligned}$$

where in the last line we used the reduced form of the Euler-Lagrange equation, i.e. the equivalent to Eq. 3 (or rather a summed version of it) noting that the Lagrangian L does not contain t. Accordingly,

$T+U=E$ where E is a constant ($=-C$). We note that in line 6 we use the fact that $\dot{q}_j^2$ is independent of $\dot{q}_i$ unless $i=j$ plus the fact that U is independent of $\dot{q}_i$.
Now, $T+U$ is the total energy (i.e. kinetic plus potential) and hence the obtained result means that the energy of such a system is conserved.[114]

6. Show that for a simple system made of a single particle of mass m in a conservative force field, Lagrange equations are equivalent to Newton's second law.
Answer: We have:

$$L\left(t,x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3\right)=L\left(t,\mathbf{r},\dot{\mathbf{r}}\right)=T-U=\frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2)-U=\frac{1}{2}m\left|\dot{\mathbf{r}}\right|^2-U$$

where $\mathbf{r}=(x_1,x_2,x_3)$ is the position vector of the particle, $\dot{\mathbf{r}}=(\dot{x}_1,\dot{x}_2,\dot{x}_3)$ is its velocity vector and $U=U(x_1,x_2,x_3)=U(\mathbf{r})$. Accordingly, the three Lagrangian equations are $\frac{\partial L}{\partial x_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)=0$ ($i=1,2,3$), that is:

$$\begin{aligned}
-\frac{\partial U}{\partial x_i}-\frac{d}{dt}\left(m\dot{x}_i\right) &= 0 \qquad\qquad (i=1,2,3)\\
-\frac{\partial U}{\partial x_i}-m\ddot{x}_i &= 0\\
-\frac{\partial U}{\partial x_i} &= m\ddot{x}_i\\
-\nabla U &= m\ddot{\mathbf{r}}\\
\mathbf{F} &= m\ddot{\mathbf{r}}
\end{aligned}$$

where in the last line we used the fact that the conservative force is the negative gradient of the potential energy. As we see, the last line is Newton's second law (i.e. force $\mathbf{F}$ equals mass m times acceleration $\ddot{\mathbf{r}}$).

7. Find the Hamiltonian formulation[115] of a simple mechanical system consisting of a free particle of mass m moving in a plane. Also, interpret the result.
Answer: We use polar coordinates (ρ,ϕ) which are sufficient to describe the motion of this particle whose movement is restricted to a plane. Now, the particle is free and hence its potential energy U is zero.[116] Regarding its kinetic energy, it is $T=\frac{1}{2}m\left(\dot{\rho}^2+\rho^2\dot{\phi}^2\right)$ due to the fact that the velocity has in general radial and azimuthal components. Hence, the Lagrangian of this system is:

$$L\left(t,\rho,\phi,\dot{\rho},\dot{\phi}\right)=T-U=\frac{1}{2}m\left(\dot{\rho}^2+\rho^2\dot{\phi}^2\right)$$

Accordingly, we have two Euler-Lagrange equations (one equation for each coordinate):

$$\begin{aligned}
\frac{\partial L}{\partial \rho}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) &= 0\\
\frac{\partial}{\partial \rho}\left[\frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2\right]-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\rho}}\left[\frac{1}{2}m\dot{\rho}^2+\frac{1}{2}m\rho^2\dot{\phi}^2\right]\right) &= 0\\
m\rho\dot{\phi}^2-\frac{d}{dt}\left(m\dot{\rho}\right) &= 0\\
m\rho\dot{\phi}^2-m\ddot{\rho} &= 0 \qquad (120)
\end{aligned}$$

AND

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[114] More technically, the total energy of a mechanical system in a conservative force field is constant along its trajectory in space-time. Alternatively, in a mechanical system whose Lagrangian is independent (explicitly) of time the particles follow trajectories along which the total energy (i.e. kinetic plus potential) of the system is conserved.

[115] We mean by "Hamiltonian formulation" in this sort of Problems the Lagrangian L and the Euler-Lagrange equation(s).

[116] In fact, it is constant that can be set to zero due to the arbitrariness of the reference level. Anyway, this may affect the Lagrangian but not the Euler-Lagrange equations.

$$\begin{aligned}
\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= 0 \\
\frac{\partial}{\partial \phi}\left[\frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\phi}}\left[\frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2\right]\right) &= 0 \\
0 - \frac{d}{dt}\left(m\rho^2\dot{\phi}\right) &= 0 \\
\frac{d}{dt}\left(m\rho^2\dot{\phi}\right) &= 0 \qquad (121)
\end{aligned}$$

Regarding the interpretation of the result, Eq. 120 represents Newton's second law (i.e. force in the radial direction equals mass times centripetal/centrifugal acceleration)[117] while Eq. 121 represents the conservation of angular momentum (noting that Eq. 121 leads to $m\rho^2\dot{\phi} = \text{constant}$).

Note: the expression of the kinetic energy T that we used above can be easily obtained (using the equivalent Cartesian system) as follows:

$$\begin{aligned}
T &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \\
&= \frac{1}{2}m\left\{\left[\frac{d}{dt}(\rho\cos\phi)\right]^2 + \left[\frac{d}{dt}(\rho\sin\phi)\right]^2\right\} \\
&= \frac{1}{2}m\left\{\left[\dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi\right]^2 + \left[\dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi\right]^2\right\} \\
&= \frac{1}{2}m\Big[\dot{\rho}^2\cos^2\phi - 2\rho\dot{\rho}\dot{\phi}\cos\phi\sin\phi + \rho^2\dot{\phi}^2\sin^2\phi + \\
&\qquad \dot{\rho}^2\sin^2\phi + 2\rho\dot{\rho}\dot{\phi}\cos\phi\sin\phi + \rho^2\dot{\phi}^2\cos^2\phi\Big] \\
&= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right)
\end{aligned}$$

8. Find the Hamiltonian formulation of a simple mechanical system consisting of a single particle of mass m in the gravitational field of the Earth.

   Answer: If q_1, q_2, q_3 stand for the Cartesian coordinates x, y, z of the particle and $\Phi(x, y, z)$ represents the gravitational potential of the Earth (noting that this potential depends on the coordinates only as indicated by the notation) then we have:

$$L\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}\right) = T - U = \frac{1}{2}mv^2 - m\Phi = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - m\Phi$$

   Accordingly, we have three Euler-Lagrange equations (one equation for each coordinate):

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \qquad \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \qquad \frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0$$

   that is:

$$-m\frac{\partial \Phi}{\partial x} - \frac{d}{dt}\left(m\dot{x}\right) = 0 \qquad -m\frac{\partial \Phi}{\partial y} - \frac{d}{dt}\left(m\dot{y}\right) = 0 \qquad -m\frac{\partial \Phi}{\partial z} - \frac{d}{dt}\left(m\dot{z}\right) = 0$$

   These equations can be simplified to:

$$\ddot{x} = -\frac{\partial \Phi}{\partial x} \qquad \ddot{y} = -\frac{\partial \Phi}{\partial y} \qquad \ddot{z} = -\frac{\partial \Phi}{\partial z}$$

---

[117] This should not contradict the fact that the particle is free.

9. Find the Hamiltonian formulation of a mechanical system consisting of a satellite of mass m orbiting the Earth. Use a normalized spherical coordinate system centered on the center of the Earth.
Answer: In this Problem q_1, q_2, q_3 stand for the spherical coordinates r, θ, ϕ of the satellite and $\Phi(r, \theta, \phi)$ represents the gravitational potential of the Earth (which solely depends on the coordinates). Now, in spherical coordinates the velocity is $\mathbf{v} = (\dot{r}, r\dot{\theta}, r\dot{\phi}\sin\theta)$ and hence $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta$ while the gravitational potential is $\Phi = -C/r$ (with C being a positive constant).[118] So, the Lagrangian of this system is:
$$L\left(t, r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}\right) = T - U = \frac{1}{2}mv^2 - m\Phi = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) + m\frac{C}{r}$$
Accordingly, we have three Euler-Lagrange equations (one equation for each coordinate):
$$\begin{aligned}
\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= 0 \qquad \Rightarrow \qquad r\dot{\theta}^2 + r\dot{\phi}^2\sin^2\theta - \frac{C}{r^2} - \frac{d\dot{r}}{dt} = 0 && (122)\\
\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= 0 \qquad \Rightarrow \qquad r^2\dot{\phi}^2\sin\theta\cos\theta - \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 && (123)\\
\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= 0 \qquad \Rightarrow \qquad \frac{d}{dt}\left(r^2\dot{\phi}\sin^2\theta\right) = 0 && (124)
\end{aligned}$$

10. Simplify the Hamiltonian formulation of Problem 9 by assuming that the orbit is in the equatorial plane of the coordinate system (i.e. the plane $\theta = \pi/2$ that passes through the center of the Earth).
Answer: In the equatorial plane $\theta = \pi/2$ and hence $\cos\theta = \dot{\theta} = 0$ and $\sin\theta = 1$. Therefore, Eqs. 122-124 reduce to:
$$\begin{aligned}
r\dot{\phi}^2 - \frac{C}{r^2} - \ddot{r} &= 0\\
0 &= 0\\
\frac{d}{dt}\left(r^2\dot{\phi}\right) &= 0
\end{aligned}$$
On discarding the second equation (which is trivial) we get two Euler-Lagrange equations:
$$\begin{aligned}
r\dot{\phi}^2 - \frac{C}{r^2} - \ddot{r} &= 0 && (125)\\
\frac{d}{dt}\left(r^2\dot{\phi}\right) &= 0 && (126)
\end{aligned}$$

11. Analyze the results of Problem 10.
Answer: If we multiply Eq. 125 with m (which we discarded earlier for simplicity) and rearrange the terms we get:
$$\frac{mC}{r^2} = m\left(r\dot{\phi}^2 - \ddot{r}\right) \qquad (127)$$
where $\frac{mC}{r^2}$ is the magnitude of the gravitational force while $(r\dot{\phi}^2 - \ddot{r})$ is the magnitude of the radial acceleration and hence Eq. 125 is just Newton's second law for the gravitational field (which is a central force field).
Regarding Eq. 126, it can be easily integrated to obtain $r^2\dot{\phi} = C_1$ (with C_1 being a constant). Now, $r^2\dot{\phi}$ is the magnitude of the angular momentum per unit mass and hence Eq. 126 represents the conservation of angular momentum. In fact, $r^2\dot{\phi}$ also represents twice the areal speed and hence the equation $r^2\dot{\phi} = C_1$ represents Kepler's second law.
Note: we may put Eq. 127 in the following form:
$$\frac{mC}{r^2} + m\ddot{r} = mr\dot{\phi}^2$$

---

[118] In fact, $C = GM$ where G is the gravitational constant and M is the mass of the Earth.

where the left hand side represents the total central force (i.e. gravitational plus inertial) while the right hand side represents mass times centripetal acceleration (and hence the equation represents Newton's second law for central gravitational fields).

12. Obtain the Hamiltonian formulation of a mechanical system consisting of a single particle of mass m executing a 1D simple harmonic motion. What type of solution this system has?
Answer: For simple harmonic motion we have:

$$L\left(t,x,\dot{x}\right)=T-U=\frac{1}{2}m\dot{x}^{2}-\frac{1}{2}kx^{2}$$

where t is time, x is the displacement of the particle from its equilibrium position, $\dot{x}$ is its speed (i.e. dx/dt) and k is a positive constant (i.e. the "spring" constant). Accordingly, we have a single Euler-Lagrange equation, that is:

$$\begin{aligned}\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) &= 0\\ \frac{\partial}{\partial x}\left[\frac{1}{2}m\dot{x}^{2}-\frac{1}{2}kx^{2}\right]-\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}}\left[\frac{1}{2}m\dot{x}^{2}-\frac{1}{2}kx^{2}\right]\right) &= 0\\ -kx-\frac{d}{dt}\left(m\dot{x}\right) &= 0\\ m\ddot{x}+kx &= 0\end{aligned}$$

The solution is obviously a sinusoidal function of time (with frequency $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$ and appropriate magnitude and phase shift).

13. Obtain the Hamiltonian formulation of a mechanical system consisting of three particles connected in series by three springs in-between (and the entire system is connected to a fixed support S). Assume that the springs are linear (Hookean) and massless and the only forces acting on the system are the spring forces (so the effect of other forces like gravity is negligible).

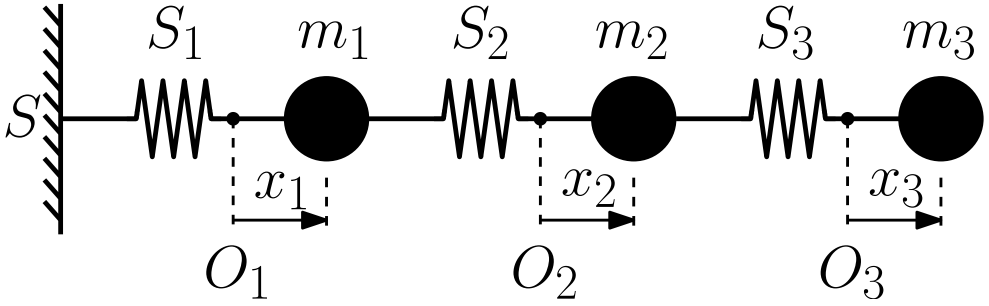

Figure 65: A schematic illustration of a system consisting of three particles (of masses m_1, m_2, m_3) connected in series to three springs S_1, S_2, S_3 and secured to a fixed support S. See Problem 13 of § 6.

Answer: Let m_1, m_2, m_3 be the masses of the particles, S_1, S_2, S_3 the springs, k_1, k_2, k_3 their spring constants, and x_1, x_2, x_3 the displacements of the particles from their equilibrium positions O_1, O_2, O_3 (see Figure 65). Now, m_1 is displaced from its equilibrium position O_1 by x_1 and hence its potential energy (which is due to the stretch/compression in S_1) is $\frac{1}{2}k_1x_1^2$. Regarding m_2, it is displaced from its equilibrium position O_2 by x_2 but part of this displacement is due to the displacement x_1 and hence its potential energy (which is due to the stretch/compression in S_2) is $\frac{1}{2}k_2(x_2-x_1)^2$. Similarly, m_3 is displaced from its equilibrium position O_3 by x_3 but part of this displacement is due to the displacement x_2 and hence its potential energy (which is due to the stretch/compression in S_3) is $\frac{1}{2}k_3(x_3-x_2)^2$. So, the potential energy of the system is $U=\frac{1}{2}k_1x_1^2+\frac{1}{2}k_2(x_2-x_1)^2+\frac{1}{2}k_3(x_3-x_2)^2$. Regarding the kinetic energy, each particle m_i $(i=1,2,3)$ has a speed $\dot{x}_i\equiv\frac{dx_i}{dt}$ which represents the temporal rate of change of its position relative to its equilibrium position (noting that all the equilibrium positions are at rest). Therefore, the kinetic energy of each particle is $\frac{1}{2}m_i\dot{x}_i^2$ and hence the kinetic energy of the system is $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2$. So, the Lagrangian of the system is:

$$L\left(t, x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3\right) = T - U = \frac{1}{2}\left[m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2 - k_1x_1^2 - k_2(x_2 - x_1)^2 - k_3(x_3 - x_2)^2\right]$$

Accordingly, we have three Euler-Lagrange equations (corresponding to $x_1, x_{2,}x_3$):

$$\begin{aligned}
\frac{\partial L}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) &= 0 &\quad\Rightarrow\quad& -k_1x_1 + k_2(x_2 - x_1) - m_1\ddot{x}_1 = 0\\
\frac{\partial L}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) &= 0 &\quad\Rightarrow\quad& -k_2(x_2 - x_1) + k_3(x_3 - x_2) - m_2\ddot{x}_2 = 0\\
\frac{\partial L}{\partial x_3} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) &= 0 &\quad\Rightarrow\quad& -k_3(x_3 - x_2) - m_3\ddot{x}_3 = 0
\end{aligned}$$

14. Obtain the Hamiltonian formulation of a simple pendulum consisting of a particle of mass m hanging at the end of an inextensible and weightless string of length l and swinging in a vertical plane where (uniform) gravity is the only active force (see Figure 66). Suggest a simple solution.

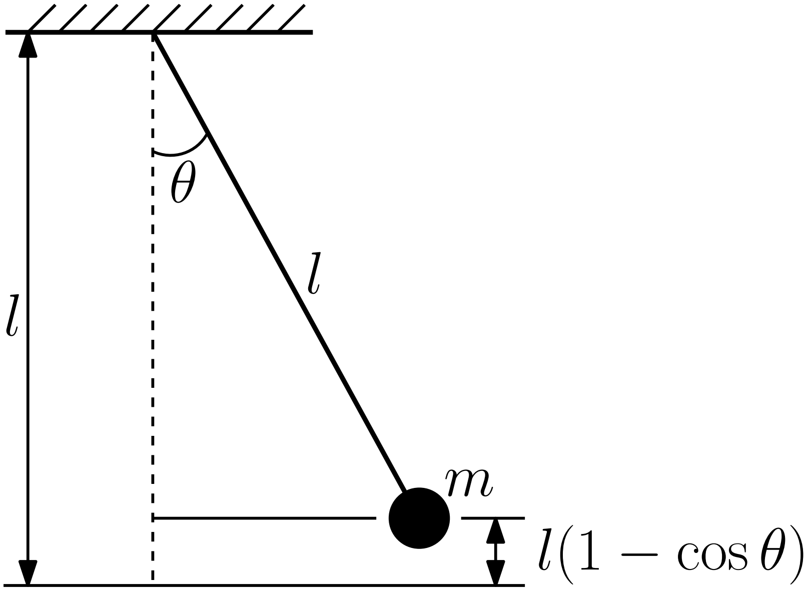

Figure 66: A schematic illustration of the simple pendulum system. See Problem 14 of § 6.

Answer: For this system we have:

$$L\left(t, \theta, \dot{\theta}\right) = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\left(1 - \cos\theta\right)$$

where t is time, θ is the angle of displacement from equilibrium, $\dot{\theta}$ is the angular speed (i.e. $d\theta/dt$), and g is the magnitude of the gravitational field (the so-called gravitational acceleration).[119] Accordingly, we have a single Euler-Lagrange equation, that is:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

---

[119] The zero potential energy reference corresponds to the position $\theta = 0$, i.e. when m is below the ceiling by a distance l.

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left[\frac{1}{2}ml^2\dot{\theta}^2 - mgl\left(1-\cos\theta\right)\right] - \frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}\left[\frac{1}{2}ml^2\dot{\theta}^2 - mgl\left(1-\cos\theta\right)\right]\right) &= 0 \\
-mgl\sin\theta - \frac{d}{dt}\left(ml^2\dot{\theta}\right) &= 0 \\
-mgl\sin\theta - ml^2\ddot{\theta} &= 0 \\
l\ddot{\theta} + g\sin\theta &= 0
\end{aligned}$$

For small θ we can use the approximation $\sin\theta \simeq \theta$ and hence the pendulum equation becomes $l\ddot{\theta} + g\theta = 0$. So, the solution (according to this simplification) is a sinusoidal function of time (with frequency $\frac{1}{2\pi}\sqrt{\frac{g}{l}}$).

15. Obtain the Hamiltonian formulation of a compound pendulum consisting of a particle of mass m_1 (hanging at the end of an inextensible and weightless string of length l_1) to which a second particle of mass m_2 (hanging at the end of an inextensible and weightless string of length l_2) is attached (see Figure 67). Again, the swing of the compound pendulum is restricted to a vertical plane and (uniform) gravity is the only active force.

Answer: Let solve this Problem using the setting of Figure 67 where we use an inverted Cartesian coordinate system (i.e. its y axis is pointing downward) in the vertical plane. The origin of the coordinate system is taken at the hanging point of the string l_1 while the zero potential energy reference is taken at $y = 0$. In this setting θ_1 and θ_2 are the angles of displacement of m_1 and m_2 from the vertical line. Now, the positions of m_1 and m_2 are:

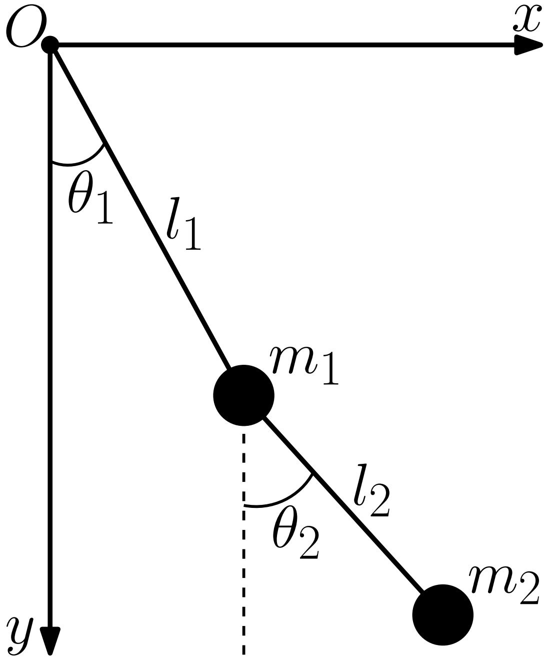

Figure 67: A schematic illustration of the compound pendulum system. See Problem 15 of § 6.

$$(x_1, y_1) = (l_1\sin\theta_1,\ l_1\cos\theta_1) \qquad \text{and} \qquad (x_2, y_2) = (l_1\sin\theta_1 + l_2\sin\theta_2,\ l_1\cos\theta_1 + l_2\cos\theta_2)$$

Hence, the potential energy of the system is:

$$\begin{aligned}
U &= -m_1 g y_1 - m_2 g y_2 \\
&= -m_1 g l_1\cos\theta_1 - m_2 g\left(l_1\cos\theta_1 + l_2\cos\theta_2\right) \\
&= \left(-m_1 g l_1 - m_2 g l_1\right)\cos\theta_1 - m_2 g l_2\cos\theta_2
\end{aligned}$$

$$= -\left[gl_1\left(m_1+m_2\right)\cos\theta_1+m_2gl_2\cos\theta_2\right]$$

while its kinetic energy is:

$$\begin{aligned}
T &= \frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2\\
&= \frac{1}{2}m_1\left(\dot{x}_1^2+\dot{y}_1^2\right)+\frac{1}{2}m_2\left(\dot{x}_2^2+\dot{y}_2^2\right)\\
&= \frac{1}{2}m_1\left(l_1^2\dot{\theta}_1^2\cos^2\theta_1+l_1^2\dot{\theta}_1^2\sin^2\theta_1\right)+\\
&\quad \frac{1}{2}m_2\left(\left[l_1\dot{\theta}_1\cos\theta_1+l_2\dot{\theta}_2\cos\theta_2\right]^2+\left[-l_1\dot{\theta}_1\sin\theta_1-l_2\dot{\theta}_2\sin\theta_2\right]^2\right)\\
&= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\Bigg(l_1^2\dot{\theta}_1^2\cos^2\theta_1+2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2+l_2^2\dot{\theta}_2^2\cos^2\theta_2+\\
&\qquad l_1^2\dot{\theta}_1^2\sin^2\theta_1+2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2+l_2^2\dot{\theta}_2^2\sin^2\theta_2\Bigg)\\
&= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2+l_2^2\dot{\theta}_2^2+2l_1l_2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\right)\right]\\
&= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2+l_2^2\dot{\theta}_2^2+2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)\right]\\
&= \frac{1}{2}\left(m_1+m_2\right)l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2l_2^2\dot{\theta}_2^2+m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)
\end{aligned}$$

So, the Lagrangian of the system is:

$$\begin{aligned}
L\left(t,\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2\right) &= T-U\\
&= \frac{1}{2}\left(m_1+m_2\right)l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2l_2^2\dot{\theta}_2^2+m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)+\\
&\quad gl_1\left(m_1+m_2\right)\cos\theta_1+m_2gl_2\cos\theta_2
\end{aligned}$$

Accordingly, we have two Euler-Lagrange equations (one equation for each θ):

$$\begin{aligned}
\frac{\partial L}{\partial\theta_1}-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) &= 0\\
-m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1-\theta_2\right)-gl_1\left(m_1+m_2\right)\sin\theta_1\\
-\frac{d}{dt}\left[\left(m_1+m_2\right)l_1^2\dot{\theta}_1+m_2l_1l_2\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)\right] &= 0\\
-m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1-\theta_2\right)-gl_1\left(m_1+m_2\right)\sin\theta_1\\
-\left(m_1+m_2\right)l_1^2\ddot{\theta}_1-m_2l_1l_2\ddot{\theta}_2\cos\left(\theta_1-\theta_2\right)+m_2l_1l_2\dot{\theta}_2\left(\dot{\theta}_1-\dot{\theta}_2\right)\sin\left(\theta_1-\theta_2\right) &= 0\\
&\text{AND}\\
\frac{\partial L}{\partial\theta_2}-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) &= 0\\
m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1-\theta_2\right)-m_2gl_2\sin\theta_2\\
-\frac{d}{dt}\left[m_2l_2^2\dot{\theta}_2+m_2l_1l_2\dot{\theta}_1\cos\left(\theta_1-\theta_2\right)\right] &= 0\\
m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1-\theta_2\right)-m_2gl_2\sin\theta_2\\
-m_2l_2^2\ddot{\theta}_2-m_2l_1l_2\ddot{\theta}_1\cos\left(\theta_1-\theta_2\right)+m_2l_1l_2\dot{\theta}_1\left(\dot{\theta}_1-\dot{\theta}_2\right)\sin\left(\theta_1-\theta_2\right) &= 0
\end{aligned}$$

16. Obtain the Hamiltonian formulation of a spherical pendulum consisting of a particle of mass m hanging at the end of an inextensible and weightless string of length l and swinging around (in all directions but restricted by the length l) where the (uniform) gravity of the Earth is the only active force.
Answer: We use a normalized spherical coordinate system centered on the fixed end of the string with the $\theta = 0$ axis (corresponding to the z axis) pointing toward the Earth center. In this Problem the dependent variables are the spherical coordinates r, θ, ϕ of the particle m and we take the zero potential energy reference to be the plane passing through the origin of coordinates and parallel to the surface of the Earth. Now, in spherical coordinates the velocity is $\mathbf{v} = (\dot{r}, r\dot{\theta}, r\dot{\phi}\sin\theta)$ but since m is always at distance l from the origin then $r = l$ and $\dot{r} = 0$ and hence the velocity of the particle is $\mathbf{v} = (0, l\dot{\theta}, l\dot{\phi}\sin\theta)$. Accordingly, the kinetic energy of the particle is $T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\dot{\phi}^2\sin^2\theta\right)$ and its potential energy is $U = -mgl\cos\theta$. Therefore, the Lagrangian of this mechanical system is:

$$L\left(t, r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}\right) = T - U = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta$$

Accordingly, we have three Euler-Lagrange equations (one equation for each coordinate):

$$\begin{aligned}
\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= 0\\
\frac{\partial}{\partial r}\left[\frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}}\left[\frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta\right]\right) &= 0\\
0 - 0 &= 0\\
0 &= 0\\
&\text{AND}\\
\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= 0\\
\frac{\partial}{\partial \theta}\left[\frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta\right]\right) &= 0\\
\frac{1}{2}ml^2\left(2\dot{\phi}^2\sin\theta\cos\theta\right) - mgl\sin\theta - \frac{d}{dt}\left[\frac{1}{2}ml^2\left(2\dot{\theta}\right)\right] &= 0\\
ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta - ml^2\ddot{\theta} &= 0\\
l\dot{\phi}^2\sin\theta\cos\theta - g\sin\theta - l\ddot{\theta} &= 0\\
&\text{AND}\\
\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= 0\\
\frac{\partial}{\partial \phi}\left[\frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\phi}}\left[\frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta\right]\right) &= 0\\
0 - \frac{d}{dt}\left(\left[\frac{1}{2}ml^2\left(2\dot{\phi}\sin^2\theta\right)\right]\right) &= 0\\
ml^2\left(\ddot{\phi}\sin^2\theta + 2\dot{\phi}\dot{\theta}\sin\theta\cos\theta\right) &= 0\\
\ddot{\phi}\sin^2\theta + 2\dot{\phi}\dot{\theta}\sin\theta\cos\theta &= 0
\end{aligned}$$

So, we have only two useful (non-trivial) equations.

17. Use a Hamiltonian approach to analyze a free system made of two massive particles interacting by a conservative force that solely depends on their separation.
Answer: If the masses of the particles are m_1 and m_2 and their position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$, then the Lagrangian of the system is:

$$L\left(t, \mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2\right) = \frac{1}{2}m_1\left|\dot{\mathbf{r}}_1\right|^2 + \frac{1}{2}m_2\left|\dot{\mathbf{r}}_2\right|^2 - U\left(\mathbf{r}\right)$$

where $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2 = (x_1, x_2, x_3)$ is the separation vector. Now, let $\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} = (X_1, X_2, X_3)$ be the position vector of the center of mass of the system (with M being the total mass, i.e. $M = m_1 + m_2$). Accordingly:

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{r}_1 + \frac{m_2}{M}\mathbf{r}_2 - \frac{m_2}{M}\mathbf{r}_2 \\
&= \frac{M}{M}\mathbf{r}_1 + \frac{m_2}{M}\mathbf{r}_2 - \frac{m_2}{M}\mathbf{r}_2 \\
&= \frac{m_1 + m_2}{M}\mathbf{r}_1 + \frac{m_2}{M}\mathbf{r}_2 - \frac{m_2}{M}\mathbf{r}_2 \\
&= \frac{m_1}{M}\mathbf{r}_1 + \frac{m_2}{M}\mathbf{r}_1 + \frac{m_2}{M}\mathbf{r}_2 - \frac{m_2}{M}\mathbf{r}_2 \\
&= \frac{m_1}{M}\mathbf{r}_1 + \frac{m_2}{M}\mathbf{r}_2 + \frac{m_2}{M}\mathbf{r}_1 - \frac{m_2}{M}\mathbf{r}_2 \\
&= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} + \frac{m_2}{M}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= \mathbf{R} + \frac{m_2}{M}\mathbf{r} \\
&\text{AND} \\
\mathbf{r}_2 &= \frac{m_1}{M}\mathbf{r}_1 - \frac{m_1}{M}\mathbf{r}_1 + \mathbf{r}_2 \\
&= \frac{m_1}{M}\mathbf{r}_1 - \frac{m_1}{M}\mathbf{r}_1 + \frac{M}{M}\mathbf{r}_2 \\
&= \frac{m_1}{M}\mathbf{r}_1 - \frac{m_1}{M}\mathbf{r}_1 + \frac{m_1 + m_2}{M}\mathbf{r}_2 \\
&= \frac{m_1}{M}\mathbf{r}_1 - \frac{m_1}{M}\mathbf{r}_1 + \frac{m_1}{M}\mathbf{r}_2 + \frac{m_2}{M}\mathbf{r}_2 \\
&= \frac{m_1}{M}\mathbf{r}_1 + \frac{m_2}{M}\mathbf{r}_2 - \frac{m_1}{M}\mathbf{r}_1 + \frac{m_1}{M}\mathbf{r}_2 \\
&= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} - \frac{m_1}{M}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= \mathbf{R} - \frac{m_1}{M}\mathbf{r}
\end{aligned}$$

Hence, $\dot{\mathbf{r}}_1 = \dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}$ and $\dot{\mathbf{r}}_2 = \dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}$. Therefore:

$$\begin{aligned}
|\dot{\mathbf{r}}_1|^2 &= \left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right) \cdot \left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right) = \left|\dot{\mathbf{R}}\right|^2 + 2\frac{m_2}{M}\dot{\mathbf{R}} \cdot \dot{\mathbf{r}} + \frac{m_2^2}{M^2}|\dot{\mathbf{r}}|^2 \\
|\dot{\mathbf{r}}_2|^2 &= \left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right) \cdot \left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right) = \left|\dot{\mathbf{R}}\right|^2 - 2\frac{m_1}{M}\dot{\mathbf{R}} \cdot \dot{\mathbf{r}} + \frac{m_1^2}{M^2}|\dot{\mathbf{r}}|^2
\end{aligned}$$

Accordingly, the Lagrangian becomes:

$$\begin{aligned}
L &= \frac{1}{2}m_1\left(\left|\dot{\mathbf{R}}\right|^2 + 2\frac{m_2}{M}\dot{\mathbf{R}} \cdot \dot{\mathbf{r}} + \frac{m_2^2}{M^2}|\dot{\mathbf{r}}|^2\right) + \frac{1}{2}m_2\left(\left|\dot{\mathbf{R}}\right|^2 - 2\frac{m_1}{M}\dot{\mathbf{R}} \cdot \dot{\mathbf{r}} + \frac{m_1^2}{M^2}|\dot{\mathbf{r}}|^2\right) - U(\mathbf{r}) \\
&= \frac{1}{2}m_1\left|\dot{\mathbf{R}}\right|^2 + \frac{1}{2}\frac{m_1m_2^2}{M^2}|\dot{\mathbf{r}}|^2 + \frac{1}{2}m_2\left|\dot{\mathbf{R}}\right|^2 + \frac{1}{2}\frac{m_2m_1^2}{M^2}|\dot{\mathbf{r}}|^2 - U(\mathbf{r}) \\
&= \frac{1}{2}(m_1 + m_2)\left|\dot{\mathbf{R}}\right|^2 + \frac{1}{2}\left(\frac{m_1m_2^2}{M^2} + \frac{m_2m_1^2}{M^2}\right)|\dot{\mathbf{r}}|^2 - U(\mathbf{r}) \\
&= \frac{1}{2}M\left|\dot{\mathbf{R}}\right|^2 + \frac{m_1m_2M}{2M^2}|\dot{\mathbf{r}}|^2 - U(\mathbf{r}) \\
&= \frac{1}{2}M\left|\dot{\mathbf{R}}\right|^2 + \frac{m_1m_2}{2M}|\dot{\mathbf{r}}|^2 - U(\mathbf{r}) \\
&= \frac{1}{2}M\left(\dot{X}_1^2 + \dot{X}_2^2 + \dot{X}_3^2\right) + \frac{m_1m_2}{2M}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - U(x_1, x_2, x_3)
\end{aligned}$$

Now, from the perspective of the center of mass (where the Lagrangian is a function of the coordinates X_1, X_2, X_3 and their derivatives as well as time), the Euler-Lagrange equations are:

$$\begin{aligned}
\frac{\partial L}{\partial X_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_i}\right) &= 0 \qquad\qquad (i = 1, 2, 3) \\
0 - \frac{d}{dt}\left(M\dot{X}_i\right) &= 0 \\
M\ddot{X}_i &= 0 \\
\ddot{X}_i &= 0 \\
\ddot{\mathbf{R}} &= \mathbf{0}
\end{aligned}$$

This result is logical because the system as a whole (and hence its center of mass) is free of any force and therefore its acceleration should vanish according to Newton's first law.

From the perspective of the interacting particles (where the Lagrangian is a function of the coordinates x_1, x_2, x_3 and their derivatives as well as time), the Euler-Lagrange equations are:

$$\begin{aligned}
\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) &= 0 \qquad\qquad (i = 1, 2, 3) \\
-\frac{\partial U}{\partial x_i} - \frac{d}{dt}\left(\frac{m_1 m_2}{M}\dot{x}_i\right) &= 0 \\
-\frac{\partial U}{\partial x_i} - \frac{m_1 m_2}{M}\ddot{x}_i &= 0 \\
\frac{m_1 m_2}{M}\ddot{x}_i &= -\frac{\partial U}{\partial x_i} \\
m_r \ddot{\mathbf{r}} &= -\nabla U
\end{aligned}$$

where m_r is the reduced mass and ∇ is the gradient operator. This result is also logical because it represents Newton's second law, i.e. mass times acceleration equals force (noting that the conservative force is the negative gradient of the potential energy).

18. Show that a particle whose trajectory in a 3D Euclidean space is restricted to an equipotential surface follows a geodesic path.

Answer: We describe the particle trajectory (which connects its start and end points) by Cartesian coordinates x, y, z. Now, since the trajectory is restricted to an equipotential surface then the potential energy of the particle is constant, that is $U = C$. Hence the Lagrangian is:

$$L\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}\right) = T - U = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - C$$

where m is the mass of the particle. Accordingly, we have three Euler-Lagrange equations (one equation for each coordinate):

$$\begin{aligned}
\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= 0 \qquad \Rightarrow \qquad \ddot{x} = 0 \\
\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) &= 0 \qquad \Rightarrow \qquad \ddot{y} = 0 \\
\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) &= 0 \qquad \Rightarrow \qquad \ddot{z} = 0
\end{aligned}$$

Now, let parameterize the Cartesian coordinates of the geodesic that connects the two end points of the trajectory with t and hence $x = x(t)$, $y = y(t)$ and $z = z(t)$. Accordingly, the geodesic is the path of optimal length s as given by:

$$s = \int_\Gamma ds = \int_\Gamma \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \int_{t_A}^{t_B} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt \equiv I\left[x, y, z\right]$$

and therefore $F = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. Now, referring to footnote [12] F is independent of t, x, y, z and hence the Euler-Lagrange equations are $\dot{x}$ = constant, $\dot{y}$ = constant and $\dot{z}$ = constant, which lead to $\ddot{x} = 0$, $\ddot{y} = 0$ and $\ddot{z} = 0$. This means that the mathematical formulation of the two problems (i.e. motion over an equipotential surface and motion along a geodesic curve) is the same and hence the trajectory as found from the Hamiltonian formulation is identical to the trajectory as found from the geodesic formulation, i.e. the particle follows a geodesic path.

Note: those who may not feel comfortable with the use of footnote [12] may use Eq. 3, that is:

$$\begin{array}{rclcl}
\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \dot{x}\dfrac{\partial\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{\partial\dot{x}} & = & C_1 & \Rightarrow & \dfrac{\dot{y}^2 + \dot{z}^2}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_1 \\
\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \dot{y}\dfrac{\partial\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{\partial\dot{y}} & = & C_2 & \Rightarrow & \dfrac{\dot{x}^2 + \dot{z}^2}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_2 \\
\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \dot{z}\dfrac{\partial\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{\partial\dot{z}} & = & C_3 & \Rightarrow & \dfrac{\dot{x}^2 + \dot{y}^2}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_3
\end{array}$$

which lead (by adding and simplifying) to $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ = constant. Hence, from Eq. 4 we get:

$$\frac{\partial F}{\partial\dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_4 \qquad \frac{\partial F}{\partial\dot{y}} = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_5 \qquad \frac{\partial F}{\partial\dot{z}} = \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_6$$

which lead to $\dot{x}$ = constant, $\dot{y}$ = constant and $\dot{z}$ = constant, and hence $\ddot{x} = 0$, $\ddot{y} = 0$ and $\ddot{z} = 0$.

19. Use a Hamiltonian approach to analyze the transverse oscillation of a taut string.

Answer: Let the string in its equilibrium (non-oscillating) state be along the x axis. To simplify the analysis, we assume:

- The oscillations are in the y direction only (and hence the string does not oscillate in the x and z directions).[120]
- The oscillations are small (and hence the displacement in the y direction is tiny).[121]
- The tension in the string is the only force involved (and hence forces like gravity and friction are negligible).[122]

Before we go through our detailed analysis we should note that this Problem is different from the previous Problems of this chapter in two important aspects. First, it is a continuous system problem (not a discrete system problem) because string is a continuous object (like curve) and hence it cannot be represented as discrete particle(s). Second, in the following formulation we use two independent variables and one dependent variable (unlike the common Hamiltonian problems where we usually have a single independent variable and multiple dependent variables).

Now, let τ be the tension (force) in the string and μ its linear mass density. Also, let the displacement of the string in the y direction be a function of time t and position x, i.e. $y = y(t, x)$. The kinetic energy of the string is the sum of the kinetic energies of its parts and hence it is given by the following integral (where the integral plays the role of *continuous* sum):[123]

$$T = \int_{x_1}^{x_2} \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 dx = \int_{x_1}^{x_2} \frac{\mu}{2} y_t^2 \, dx$$

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[120] In other words, the oscillations are restricted to the xy plane where each particle of the string moves along a straight line parallel to the y axis.

[121] More precisely, $\frac{\partial y}{\partial x} \ll 1$ for all x and at all times. However, this condition may be needed only for some restricted models (e.g. for constant tension and uniform mass density).

[122] In fact, non-conservative forces can be excluded by the restriction of the Hamiltonian approach (which we employ here) to conservative systems.

[123] We note that the (continuum) expression $\frac{\mu}{2} y_t^2 \, dx$ is no more than the usual (discrete) expression of the kinetic energy $\frac{1}{2}mv^2$ where $\mu \, dx$ represents m while y_t^2 represents v^2. So, $T = \int_{x_1}^{x_2} \frac{\mu}{2} y_t^2 \, dx$ is the continuous version of the discrete version $T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$. For a more technical approach, the reader is referred to the literature (see for instance Weinstock in the References).

where x_1 and x_2 are the x coordinates of the end (fixed) points of the string and $y_t = \partial y/\partial t$. Regarding the potential energy, we simply apply Hooke's law (in its continuous version) and hence:[124]

$$U = \int_{x_1}^{x_2} \frac{\tau}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx = \int_{x_1}^{x_2} \frac{\tau}{2} y_x^2 \, dx$$

where $y_x = \partial y/\partial x$. Hence the Lagrangian is:

$$L\left(t, x, y, y_t, y_x\right) = T - U = \int_{x_1}^{x_2} \frac{\mu}{2} y_t^2 \, dx - \int_{x_1}^{x_2} \frac{\tau}{2} y_x^2 \, dx = \frac{1}{2}\int_{x_1}^{x_2} \left(\mu y_t^2 - \tau y_x^2\right) dx$$

Now, if we apply Eq. 117 to our Lagrangian (noting that we have a single dependent variable y) we get:

$$I[y] = \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} \left[\frac{1}{2}\int_{x_1}^{x_2} \left(\mu y_t^2 - \tau y_x^2\right) dx\right] dt = \frac{1}{2}\int_{t_1}^{t_2}\int_{x_1}^{x_2} \left(\mu y_t^2 - \tau y_x^2\right) dx \, dt$$

As we see, this is a functional integral (with $F = \mu y_t^2 - \tau y_x^2$) of a variational problem with multiple independent variables (see Eq. 14 in § 1.6) and hence we use Eq. 15 (noting that t, x, y, y_t, y_x in our case correspond to $x_1, x_2, y, y_{x_1}, y_{x_2}$ in Eqs. 14 and 15), that is:

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right) &= 0 \\
\frac{\partial}{\partial y}\left[\mu y_t^2 - \tau y_x^2\right] - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial y_t}\left[\mu y_t^2 - \tau y_x^2\right]\right) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y_x}\left[\mu y_t^2 - \tau y_x^2\right]\right) &= 0 \\
0 - \frac{\partial}{\partial t}\left(2\mu y_t\right) - \frac{\partial}{\partial x}\left(-2\tau y_x\right) &= 0 \\
\frac{\partial}{\partial t}\left(\mu y_t\right) - \frac{\partial}{\partial x}\left(\tau y_x\right) &= 0
\end{aligned}$$

Now, if μ and τ are independent of t and x then the last equation becomes (noting the meaning of the partial derivatives with respect to the independent variables as explained in § 1.6):

$$\begin{aligned}
\mu\frac{\partial}{\partial t}\left(y_t\right) - \tau\frac{\partial}{\partial x}\left(y_x\right) &= 0 \\
\mu\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) - \tau\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) &= 0 \\
\mu\frac{\partial^2 y}{\partial t^2} - \tau\frac{\partial^2 y}{\partial x^2} &= 0
\end{aligned}$$

which is the 1D wave equation for small transverse oscillations on a taut string of uniform linear density and constant tension (with wave speed $v_w = \sqrt{\tau/\mu}$).

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[124] Instead of going through detailed analysis (which is time consuming and distracting from our variational objective) we can justify the above integral as follows (using slack notations and deliberations):

$$U = \int_{x_1}^{x_2} \frac{\tau}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx = \int_{x_1}^{x_2} \frac{\tau}{2}\frac{\partial y}{\partial x}\frac{\partial y}{\partial x} dx = \int_{x_1}^{x_2} \frac{1}{2}\frac{\tau}{\partial x}\partial y \partial y \frac{dx}{\partial x} = \int_{x_1}^{x_2} \frac{1}{2}\frac{\tau}{\partial x}\left(\partial y\right)^2$$

As we see, $\frac{\tau}{\partial x}$ corresponds to k and $(\partial y)^2$ corresponds to x^2 in the expression of the potential energy of the common form of Hooke's law (i.e. $U = \frac{1}{2}kx^2$). So, $U = \int_{x_1}^{x_2} \frac{\tau}{2} y_x^2 \, dx$ is the continuous version of the discrete version $U = \sum_{i=1}^{n} \frac{1}{2} k_i x_i^2$. For a more technical approach, the reader is referred to the literature (see for instance Weinstock in the References).

Chapter 7
Sturm-Liouville Problems

The Sturm-Liouville problems are defined by the following differential equation (on a given interval $a \le x \le b$):[125]

$$-\frac{d}{dx}\left(py'\right) + qy = \lambda wy \tag{128}$$

where p, q, w, y are functions of x (with p and w not vanishing on the interval), λ is an eigenvalue of the eigenfunction y, and $y' = dy/dx$. It can be easily shown that Sturm-Liouville problems can be formulated and solved as variational problems. In fact, the Sturm-Liouville differential equation is no more than the Euler-Lagrange equation for a certain type of variational problems with constraint (see Problem 3). This establishes a relationship between the eigenvalue problems and the calculus of variations (see Problem 7).

Problems

1. Write the Sturm-Liouville equations for the following sets of p, q, w:
   (**a**) $p = 1$, $q = x$, $w = 1$.
   (**b**) $p = a$, $q = e^x$, $w = x^b$ (where a and b are constants).
   (**c**) $p = -x$, $q = x$, $w = x^3$.
   Answer:
   (**a**) Inserting p, q, w into Eq. 128 we get:

$$-\frac{d}{dx}\left(1\,y'\right) + xy = \lambda\,1\,y \qquad \text{that is} \qquad -y'' + xy = \lambda y$$

   (**b**) Inserting p, q, w into Eq. 128 we get:

$$-\frac{d}{dx}\left(ay'\right) + e^x y = \lambda x^b y \qquad \text{that is} \qquad -ay'' + e^x y = \lambda x^b y$$

   (**c**) Inserting p, q, w into Eq. 128 we get:

$$-\frac{d}{dx}\left(-xy'\right) + xy = \lambda x^3 y \qquad \text{that is} \qquad xy'' + y' + xy = \lambda x^3 y$$

2. What are the p, q, w for the following Sturm-Liouville equations:
   (**a**) $y'' - 5x^3 y = -2\lambda xy$.
   (**b**) $x^2 y'' + 2xy' + y\ln|x| = -\alpha\lambda y$ (where α is a constant).
   (**c**) $e^{2x}y'' + 2e^{2x}y' - \frac{y}{x} = -\frac{\lambda 5y}{x^3}$.
   (**d**) $y'' - xy + \lambda x^2 y = 0$.
   Answer:
   (**a**) Putting the equation in the standard form of Sturm-Liouville equation (Eq. 128) we get:

$$-\frac{d}{dx}\left(-y'\right) + \left(-5x^3\right)y = \lambda\left(-2x\right)y$$

   By comparing this to Eq. 128 we see: $p = -1$, $q = -5x^3$, and $w = -2x$.
   (**b**) Putting the equation in the standard form of Sturm-Liouville equation (Eq. 128) we get:

$$\frac{d}{dx}\left(x^2 y'\right) + y\ln|x| \quad = \quad -\alpha\lambda y$$

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[125] The reader should note that Sturm-Liouville equation may be given in other forms which in most cases differ from this form by the sign of some terms and possibly minor differences in notation and convention.

$$-\frac{d}{dx}\left(-x^2y'\right)+\left(\ln|x|\right)y \quad = \quad \lambda\left(-\alpha\right)y$$

By comparing this to Eq. 128 we see: $p=-x^2$, $q=\ln|x|$, and $w=-\alpha$.
(**c**) Putting the equation in the standard form of Sturm-Liouville equation (Eq. 128) we get:

$$\begin{aligned}-e^{2x}y''-2e^{2x}y'+\frac{y}{x} &= \frac{\lambda 5y}{x^3}\\ -\left(e^{2x}y''+2e^{2x}y'\right)+\left(\frac{1}{x}\right)y &= \lambda\left(\frac{5}{x^3}\right)y\\ -\frac{d}{dx}\left(e^{2x}y'\right)+\left(\frac{1}{x}\right)y &= \lambda\left(\frac{5}{x^3}\right)y\end{aligned}$$

By comparing this to Eq. 128 we see: $p=e^{2x}$, $q=1/x$, and $w=5/x^3$.
(**d**) Putting the equation in the standard form of Sturm-Liouville equation (Eq. 128) we get:

$$\begin{aligned}y''-xy &= -\lambda x^2y\\ -\frac{d}{dx}\left(-y'\right)+\left(-x\right)y &= \lambda\left(-x^2\right)y\end{aligned}$$

By comparing this to Eq. 128 we see: $p=-1$, $q=-x$, and $w=-x^2$.

3. Show that the Sturm-Liouville problem (as given by Eq. 128) is equivalent to the constrained variational problem: $I[y]=I_1[y]-\lambda I_2[y]$ where:[126]

$$I_1[y]=\int_a^b\left(py'^2+qy^2\right)\,dx \qquad \text{and} \qquad I_2[y]=\int_a^b wy^2\,dx \tag{129}$$

Answer: According to the formulation of constrained variational problems (see § 1.8) we have:

$$I[y]=I_1[y]-\lambda I_2[y]=\left\{\int_a^b\left(py'^2+qy^2\right)\,dx\right\}-\lambda\left\{\int_a^b wy^2\,dx\right\}=\int_a^b\left(py'^2+qy^2-\lambda wy^2\right)\,dx \tag{130}$$

Hence, $H\equiv F-\lambda G=py'^2+qy^2-\lambda wy^2$ and the Euler-Lagrange equation (see Eq. 22) is:

$$\begin{aligned}\frac{\partial}{\partial y}\left[py'^2+qy^2-\lambda wy^2\right]-\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left[py'^2+qy^2-\lambda wy^2\right]\right) &= 0\\ 2qy-2\lambda wy-\frac{d}{dx}\left(2py'\right) &= 0\\ -2\frac{d}{dx}\left(py'\right)+2qy &= 2\lambda wy\\ -\frac{d}{dx}\left(py'\right)+qy &= \lambda wy\end{aligned}$$

which is the Sturm-Liouville equation (Eq. 128).
Note: a consequence of the result of this Problem is that Sturm-Liouville problems can be treated and formulated as constrained variational problems (see Problem 4), and some (but not all)[127] constrained variational problems can be treated and formulated as Sturm-Liouville problems (see Problem 5).

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[126] The reader should note that the eigenvalue of the Sturm-Liouville problem is represented by the Lagrange multiplier λ of the variational problem. We should also note that the minus sign in the constrained formulation is allowed because the sign of λ is rather arbitrary. The use of the minus (instead of plus) sign is to keep the given form of Sturm-Liouville equation.

[127] This restriction is due to the fact that I_1 and I_2 in Eq. 129 are restricted to certain forms.

4. Find the functional integrals I for the Sturm-Liouville equations of Problem 2.
Answer: From Eq. 130 we have $I = \int_a^b \left(py'^2 + qy^2 - \lambda wy^2\right) dx$. So, from the results of Problem 2 we have:
(**a**) $p = -1$, $q = -5x^3$, and $w = -2x$. Hence:

$$I = \int_a^b \left(-y'^2 - 5x^3y^2 + \lambda 2xy^2\right) dx$$

(**b**) $p = -x^2$, $q = \ln|x|$, and $w = -\alpha$. Hence:

$$I = \int_a^b \left(-x^2y'^2 + y^2 \ln|x| + \lambda\alpha y^2\right) dx$$

(**c**) $p = e^{2x}$, $q = 1/x$, and $w = 5/x^3$. Hence:

$$I = \int_a^b \left(e^{2x}y'^2 + \frac{1}{x}y^2 - \lambda\frac{5}{x^3}y^2\right) dx$$

(**d**) $p = -1$, $q = -x$, and $w = -x^2$. Hence:

$$I = \int_a^b \left(-y'^2 - xy^2 + \lambda x^2y^2\right) dx$$

5. Find the Euler-Lagrange equation for the constrained variational problem of part (a) of Problem 5 of § 1.8 and the constrained variational problem of part (b) of Problem 7 of § 1.8 without applying Eq. 22.
Answer: As we saw in Problem 3, H of a constrained variational problem (of a certain form whose I_1 and I_2 are given by Eq. 129) is given by $H = py'^2 + qy^2 - \lambda wy^2$ where p, q, w are the parameters of the Sturm-Liouville equation that corresponds to the constrained variational problem (and hence the Sturm-Liouville equation represents the Euler-Lagrange equation of the given constrained variational problem).
Now, for part (a) of Problem 5 of § 1.8 we have $H \equiv py'^2 + qy^2 - \lambda wy^2 = xy'^2 + \lambda x^2y^2$ and hence the Euler-Lagrange equation for this constrained variational problem is the Sturm-Liouville equation with $p = x$, $q = 0$ and $w = -x^2$, that is:

$$\begin{aligned} -\frac{d}{dx}(xy') + 0 &= -\lambda x^2 y \\ -xy'' - y' &= -\lambda x^2 y \\ xy'' + y' - \lambda x^2 y &= 0 \end{aligned}$$

which is what we found in part (a) of Problem 5 of § 1.8 by applying Eq. 22.
Similarly, for part (b) of Problem 7 of § 1.8 we have $H \equiv py'^2 + qy^2 - \lambda wy^2 = y'^2 - \lambda y^2$ and hence the Euler-Lagrange equation for this constrained variational problem is the Sturm-Liouville equation with $p = 1$, $q = 0$ and $w = 1$, that is:

$$\begin{aligned} -\frac{d}{dx}(y') + 0 &= \lambda y \\ y'' + \lambda y &= 0 \end{aligned}$$

which is what we found in part (b) of Problem 7 of § 1.8 by applying Eq. 22.

6. Using the Sturm-Liouville equation (Eq. 128), confirm that $\lambda = I_1/I_2$.
Answer: Multiplying Eq. 128 with y, we get:

$$-y\frac{d}{dx}(py') + qy^2 = \lambda wy^2$$

$$\begin{aligned}
-\int_a^b y\frac{d}{dx}\left(py'\right)dx+\int_a^b qy^2\,dx &= \int_a^b \lambda wy^2\,dx && \text{(integrating)}\\
-\left[ypy'\right]_a^b+\int_a^b py'^2\,dx+\int_a^b qy^2\,dx &= \int_a^b \lambda wy^2\,dx && \text{(integration by parts)}\\
\int_a^b py'^2\,dx+\int_a^b qy^2\,dx &= \int_a^b \lambda wy^2\,dx && \text{(using suitable boundary conditions)}\\
\int_a^b \left(py'^2+qy^2\right)dx &= \lambda\int_a^b wy^2\,dx &&\\
I_1 &= \lambda I_2 && \text{(see Eq. 129)}\\
\lambda &= \frac{I_1}{I_2} &&
\end{aligned}$$

Note: in the integration by parts formula $\int u\,dv = uv - \int v\,du$ we use $u = y$ and $v = py'$.

7. Referring to the formulation of Problem 3 and assuming normalization such that $I_2[y] = 1$,[128] show that the stationary values of $I_1[y]$ of the variational problem produce the eigenvalues of the Sturm-Liouville problem.
Answer: From the result of Problem 6 and the assumption $I_2[y] = 1$ we have:[129]

$$\lambda = \frac{I_1}{I_2} = I_1$$

Accordingly, the stationary values of I_1 of the variational problem produce the eigenvalues (λ's) of the Sturm-Liouville problem, as required.

8. Referring to the formulation of Problem 3, show that obtaining the stationary (or extreme) values of I is equivalent to obtaining the stationary (or extreme) values of I_1/I_2.[130]
Answer: Noting that variation follows the pattern of differential (and hence the rules of differentiation apply in its manipulation), we have:

$$\begin{aligned}
\delta\left(\frac{I_1}{I_2}\right) &= \frac{\delta I_1}{I_2}-\frac{I_1}{I_2^2}\delta I_2\\
&= \frac{\delta I_1}{I_2}-\frac{(I_1/I_2)}{I_2}\delta I_2\\
&= \frac{\delta I_1-(I_1/I_2)\,\delta I_2}{I_2}\\
&= \frac{\delta I_1-\lambda\delta I_2}{I_2}\\
&= \frac{\delta I}{I_2}
\end{aligned}$$

where in line 4 we use $I_1/I_2 = \lambda$ (see Problem 6). Now, since I_2 is constant (see § 1.8) the last equation means that stationarizing (or extremizing) I is equivalent to stationarizing (or extremizing) I_1/I_2.

9. Discuss and analyze the result of Problem 8.
Answer: Since stationarizing I is equivalent to stationarizing I_1/I_2 $(= \lambda)$ then any function y that stationarizes I should stationarize I_1/I_2 (and vice versa). Now, because the functions that stationarize I are solutions of the Sturm-Liouville equation (see Problem 3) then this means that obtaining the functions that stationarize I_1/I_2 is equivalent to obtaining the functions that are solutions to the

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[128] We remind the reader that according to the formulation of constrained variational problems (see § 1.8) I_2 is constant.

[129] The reader should be careful in reading and interpreting this equation and its alike. This is due mainly to the (rather loose) use of I_1 where sometimes it stands for the value and sometimes for the stationary value.

[130] Although this seems trivial (considering that $I_2[y]$ is constant), the purpose is to show this technically (with the assumption $I_2[y]$ = constant being used only in the final stage).

Sturm-Liouville equation. In fact, this should facilitate the solution of (certain types of) differential equations[131] by the methods of variational calculus (as well as benefiting from the techniques of solving differential equations in variational calculus). This will also have implications and consequences on the estimation and evaluation of the eigenvalues and eigenfunctions of the Sturm-Liouville equations (as will be demonstrated in the upcoming Problems).

10. Using the results of the previous Problems, try to propose a method for estimating the eigenvalues and eigenfunctions (and obtaining bounds and approximations) of the Sturm-Liouville problems.

Answer:[132] Based on the results of the previous Problems, the *stationary values* of I_1/I_2 of a (restricted) variational problem (that complies with the above formulations and conditions) are the eigenvalues of the corresponding Sturm-Liouville problem. This means that the *values* of I_1/I_2 for a given problem should lie between the minimum eigenvalue λ_m and the maximum eigenvalue λ_M of the Sturm-Liouville equation, that is:[133]

$$\lambda_m \le \frac{I_1}{I_2} \le \lambda_M$$

Accordingly, an estimation of I_1/I_2 for a given problem will provide an upper bound on λ_m and a lower bound on λ_M without going through any variational process or optimization procedure.[134] Moreover, a (guessed) trial function y_t that is used in this estimation could provide an approximation to the stationarizing (i.e. true) eigenfunction y that corresponds to these eigenvalues (i.e. λ_m and λ_M) where the best of the trial functions usually corresponds to the best of the estimated eigenvalues. These issues will be clarified in Problems 11 and 12.

11. Find an upper bound on the lowest eigenvalue of the following Sturm-Liouville problem:

$$y'' + \lambda y = 0 \qquad \text{with boundary conditions } y(0) = 0,\ y(1) = 1 \text{ and } y'(1) = 0$$

using the following trial functions (which all satisfy the given boundary conditions as they should be):[135]

(**a**) $y_t = x^4 + x^3 - 6x^2 + 5x$ (**b**) $y_t = x^3 - 3x^2 + 3x$ (**c**) $y_t = -x^3 + x^2 + x$
(**d**) $y_t = 2x - x^2$ (**e**) $y_t = x^4 - \frac{5}{2}x^3 + x^2 + \frac{3}{2}x$.

Answer: The equation $y'' + \lambda y = 0$ can be put in the following form: $-\frac{d}{dx}(y') = \lambda y$. Comparing this form to the standard form of Sturm-Liouville equation (as given by Eq. 128) we have:

$$p = 1 \qquad\qquad q = 0 \qquad\qquad w = 1$$

Accordingly (see Eq. 129):

$$I_1 = \int_0^1 \left(py'^2 + qy^2\right) dx = \int_0^1 y'^2\,dx \tag{131}$$

$$I_2 = \int_0^1 wy^2\,dx = \int_0^1 y^2\,dx \tag{132}$$

(**a**) For the trial function $y_t = x^4 + x^3 - 6x^2 + 5x$ we have:

$$I_1 = \int_0^1 \left(4x^3 + 3x^2 - 12x + 5\right)^2 dx = \int_0^1 \left(16x^6 + 24x^5 - 87x^4 - 32x^3 + 174x^2 - 120x + 25\right) dx$$

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[131] Noting that p, q, w in the Sturm-Liouville equation are generic and can fit a wide range of functions, the differential equations of Sturm-Liouville type are not as restricted as might be thought.

[132] We should note that other conditions and restrictions are required to make the explanations and arguments in this answer more rigorous. The details should be sought in more specialized texts on Sturm-Liouville problems and their relation to variational problems and variational calculus.

[133] Again, the reader should be careful in reading and interpreting this equation and its alike. This is due mainly to the (rather loose) use of I_1/I_2 where sometimes it stands for the stationary value (corresponding to the stationarizing/extremizing function) and sometimes for the value (corresponding to an arbitrary function which is usually an approximation to the stationarizing/extremizing function). We should also note that there are certain restrictions on λ_m and λ_M for the validity of this equation (the details should be sought in more specialized texts).

[134] In fact, with certain conditions and insights the values of I_1/I_2 may also provide approximations to the corresponding eigenvalues (as will be clarified later).

[135] The subscript t in y_t in this Problem stands for "trial" and does not symbolize partial derivative with respect to t.

$$\begin{aligned}
&= \left[\frac{16}{7}x^7 + 4x^6 - \frac{87}{5}x^5 - 8x^4 + 58x^3 - 60x^2 + 25x\right]_0^1 = \frac{136}{35} \\
I_2 &= \int_0^1 \left(x^4 + x^3 - 6x^2 + 5x\right)^2\,dx = \int_0^1 \left(x^8 + 2x^7 - 11x^6 - 2x^5 + 46x^4 - 60x^3 + 25x^2\right)\,dx \\
&= \left[\frac{1}{9}x^9 + \frac{1}{4}x^8 - \frac{11}{7}x^7 - \frac{1}{3}x^6 + \frac{46}{5}x^5 - 15x^4 + \frac{25}{3}x^3\right]_0^1 = \frac{1247}{1260}
\end{aligned}$$

Hence, an upper bound on the lowest eigenvalue is $I_1/I_2 = 4896/1247 \simeq 3.9262$.
(**b**) For the trial function $y_t = x^3 - 3x^2 + 3x$ we have:

$$\begin{aligned}
I_1 &= \int_0^1 \left(3x^2 - 6x + 3\right)^2\,dx = \int_0^1 \left(9x^4 - 36x^3 + 54x^2 - 36x + 9\right)\,dx \\
&= \left[\frac{9}{5}x^5 - 9x^4 + 18x^3 - 18x^2 + 9x\right]_0^1 = \frac{9}{5} \\
I_2 &= \int_0^1 \left(x^3 - 3x^2 + 3x\right)^2\,dx = \int_0^1 \left(x^6 - 6x^5 + 15x^4 - 18x^3 + 9x^2\right)\,dx \\
&= \left[\frac{1}{7}x^7 - x^6 + 3x^5 - \frac{9}{2}x^4 + 3x^3\right]_0^1 = \frac{9}{14}
\end{aligned}$$

Hence, an upper bound on the lowest eigenvalue is $I_1/I_2 = 14/5 = 2.8$.
(**c**) For the trial function $y_t = -x^3 + x^2 + x$ we have:

$$\begin{aligned}
I_1 &= \int_0^1 \left(-3x^2 + 2x + 1\right)^2\,dx = \int_0^1 \left(9x^4 - 12x^3 - 2x^2 + 4x + 1\right)\,dx \\
&= \left[\frac{9}{5}x^5 - 3x^4 - \frac{2}{3}x^3 + 2x^2 + x\right]_0^1 = \frac{17}{15} \\
I_2 &= \int_0^1 \left(-x^3 + x^2 + x\right)^2\,dx = \int_0^1 \left(x^6 - 2x^5 - x^4 + 2x^3 + x^2\right)\,dx \\
&= \left[\frac{1}{7}x^7 - \frac{1}{3}x^6 - \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3\right]_0^1 = \frac{31}{70}
\end{aligned}$$

Hence, an upper bound on the lowest eigenvalue is $I_1/I_2 = 238/93 \simeq 2.5591$.
(**d**) For the trial function $y_t = 2x - x^2$ we have:

$$\begin{aligned}
I_1 &= \int_0^1 (2 - 2x)^2\,dx = \int_0^1 \left(4 - 8x + 4x^2\right)\,dx = \left[4x - 4x^2 + \frac{4}{3}x^3\right]_0^1 = \frac{4}{3} \\
I_2 &= \int_0^1 \left(2x - x^2\right)^2\,dx = \int_0^1 \left(4x^2 - 4x^3 + x^4\right)\,dx = \left[\frac{4}{3}x^3 - x^4 + \frac{x^5}{5}\right]_0^1 = \frac{8}{15}
\end{aligned}$$

Hence, an upper bound on the lowest eigenvalue is $I_1/I_2 = 5/2 = 2.5$.
(**e**) For the trial function $y_t = x^4 - \frac{5}{2}x^3 + x^2 + \frac{3}{2}x$ we have:

$$\begin{aligned}
I_1 &= \int_0^1 \left(4x^3 - \frac{15}{2}x^2 + 2x + \frac{3}{2}\right)^2\,dx = \int_0^1 \left(16x^6 - 60x^5 + \frac{289}{4}x^4 - 18x^3 - \frac{37}{2}x^2 + 6x + \frac{9}{4}\right)\,dx \\
&= \left[\frac{16}{7}x^7 - 10x^6 + \frac{289}{20}x^5 - \frac{9}{2}x^4 - \frac{37}{6}x^3 + 3x^2 + \frac{9}{4}x\right]_0^1 = \frac{277}{210} \\
I_2 &= \int_0^1 \left(x^4 - \frac{5}{2}x^3 + x^2 + \frac{3}{2}x\right)^2\,dx = \int_0^1 \left(x^8 - 5x^7 + \frac{33}{4}x^6 - 2x^5 - \frac{13}{2}x^4 + 3x^3 + \frac{9}{4}x^2\right)\,dx
\end{aligned}$$

$$= \left[\frac{1}{9}x^9 - \frac{5}{8}x^8 + \frac{33}{28}x^7 - \frac{1}{3}x^6 - \frac{13}{10}x^5 + \frac{3}{4}x^4 + \frac{3}{4}x^3\right]_0^1 = \frac{1339}{2520}$$

Hence, an upper bound on the lowest eigenvalue is $I_1/I_2 = 3324/1339 \simeq 2.4825$.

12. Discuss the results of Problem 11.

Answer: Let first obtain the exact solution of the given Sturm-Liouville problem. In fact, this problem was solved in part (b) of Problem 7 of § 1.8 as a variational problem with constraint $\big[$but without the boundary condition $y(1) = 1\big]$. Now, if $\lambda \le 0$ then (according to Problem 7 of § 1.8) the solution is $y = 0$ which does not satisfy the boundary condition $y(1) = 1$ (which we imposed in Problem 11). Therefore, we should have $\lambda > 0$ and hence the solution is $y = b\sin\left(\frac{\pi x}{2}\right)$ which when combined with the boundary condition $y(1) = 1$ yields $b = 1$. So, the exact solution (eigenfunction) of the given Sturm-Liouville problem is $y = \sin\left(\frac{\pi x}{2}\right)$ with an eigenvalue $\lambda = \pi^2/4 \simeq 2.4674$ (as can be checked by substitution in the equation $y'' + \lambda y = 0$ and verifying the given boundary conditions).

Now, if we plot this solution (see Figure 68) beside the trial functions (which can be seen as approximate solutions) of Problem 11 we can see that as the trial functions become closer and closer to the exact solution (as we move from a to e) the approximations to the eigenvalue become closer and closer to the exact eigenvalue and hence the best estimation of the eigenvalue is obtained from the best approximation of the eigenfunction.

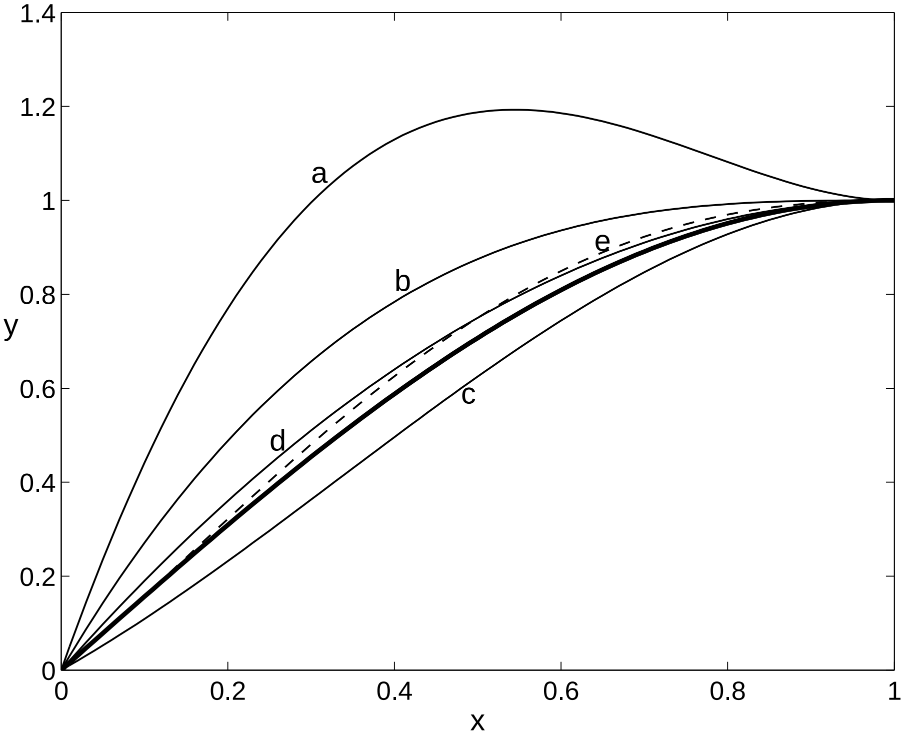

Figure 68: Plot of the exact solution $y = \sin\left(\frac{\pi x}{2}\right)$ of Problem 12 of § 7 alongside the trial functions of Problem 11 of § 7. The curve of the exact solution is solid thick while the curves of the trial functions are labeled with a, b, c, d, e (according to their labels in Problem 11 of § 7) with the curve e being dashed for clear distinction.

13. Verify the result of Problem 6 by showing that using the exact solution (i.e. eigenfunction) of Problem 12 as a trial function will produce the (exact) eigenvalue.

Answer: The exact solution is $y = \sin\left(\frac{\pi x}{2}\right)$ and its derivative is $y' = \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)$. Hence (see Eqs. 131

and 132):

$$\begin{aligned} I_1 &= \int_0^1 \left[\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\right]^2 dx = \frac{\pi^2}{4}\int_0^1 \cos^2\left(\frac{\pi x}{2}\right) dx = \frac{\pi^2}{4}\left[\frac{1}{2}\right] = \frac{\pi^2}{8} \\ I_2 &= \int_0^1 \sin^2\left(\frac{\pi x}{2}\right) dx = \frac{1}{2} \end{aligned}$$

Therefore, $I_1/I_2 = \pi^2/4$ which is the (exact) eigenvalue λ.[136]

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[136] In fact, this shows that the stationary values of I_1/I_2 of a (restricted) variational problem are the eigenvalues of the corresponding Sturm-Liouville problem (see Problem 10). It should be obvious that the "stationary value" means the value obtained from using the stationarizing (or extremizing) function.

Chapter 8
Rayleigh-Ritz Method

This is an approximation technique for finding the extremizing function in variational problems. The method can be instigated and applied numerically as well as analytically. Thanks to its flexibility, relative simplicity and natural adaptability to numerical implementations it is widely used in science and engineering. The method is based on starting from a guess representing a generic form of the extremizing function where this guess can be regarded as an approximation to the real (or exact) extremizing function which is the sought solution to the variational problem. The generic form is then developed and determined by a variational procedure (possibly in a gradual process as will be clarified later in the following remarks and in some of the upcoming Problems).

To put it in more practical terms, suppose that we are looking for an extremizing function $y = y(x)$ of an integral $I[y]$ representing the functional of a variational problem. Also assume that the function y can be approximated by a linear combination of linearly independent basis functions $\phi_i = \phi_i(x)$ $(i = 0, \cdots, n)$ of a certain type (e.g. polynomial or sinusoidal) and hence we can write:

$$y \simeq \phi_0 + \sum_{i=1}^{n} c_i \phi_i \tag{133}$$

where c_i are constants to be determined during the variational procedure.[137] So, all we need to determine y (or rather its approximation) is to determine the constants c_i's since the basis functions are known. This means that we simply converted (or rather reduced) our task from determining y itself to determining a set of constants assuming that the general form of y is known (as determined by the chosen type of the basis functions). In fact, our functional integral $I[y]$ can now be written (rather more appropriately although we will not do that) as $I[c_1, \cdots, c_i]$ because this functional is now dependent (in its variation and optimization) on the constants c_i's. In other words, this functional is extremized (or stationarized) by the set of c_i's and hence we can tackle the variational problem by seeking the solution of the system of equations:

$$\frac{\partial I}{\partial c_i} = 0 \qquad (i = 1, \cdots, n) \tag{134}$$

In fact, the best way to understand the rationale and procedure of the Rayleigh-Ritz method is to use it in some practical problems to understand and appreciate how and why it works. However, before that it is important to note the following points about this method:

(**a**) Although the guessed form (as determined by the choice of the type of the basis functions) is rather arbitrary, this form should be chosen to be as close as possible to the real form (assuming that we have an idea about the real form) so that we can get the best approximation to the real solution. In fact, if the form of the real solution is known then the guessed form should be chosen to match the real form so that the obtained approximation will be very close (and potentially identical) to the real solution. For example, if we know that y is a sinusoidal function then we should choose our basis functions to be sinusoidal so that we obtain better results.

(**b**) The Rayleigh-Ritz method as described above can be generalized and extended to include other variations and flavors such as having multiple variables. So, in this regard it is like the variational treatment in its analytical form (as represented by the Euler-Lagrange equation in its various variations and flavors which we investigated in the sections of chapter 1).

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[137] The choice of c_0 to be 1 does not affect the generality since we can always divide by $c_0 \neq 0$ (or absorb non-unity factor into ϕ_0) to reduce the form of y to the above form. In fact, this is related to the implementation of the boundary conditions as will be clarified later.

(**c**) The Rayleigh-Ritz method can be used for estimating the eigenvalues in Sturm-Liouville problems.
(**d**) Noting that the Rayleigh-Ritz method is used in boundary value problems, the zeroth basis function ϕ_0 is usually chosen to satisfy the given boundary conditions while all the other basis functions ϕ_i ($i = 1, \cdots, n$) are chosen to vanish at the boundaries.[138]
(**e**) As indicated above, the determination of the form of the extremizing function may be done in a gradual process by starting from a certain order of approximation and moving to higher orders of approximation (if necessary) where this process stops when we get the required accuracy from the most recent approximation (according to certain criteria). For example, we may start from first order approximation $y \simeq y_1 = \phi_0 + c_1\phi_1$ where we need only to determine c_1. If, we are happy with y_1 then we stop the process; otherwise we go to the second order approximation $y \simeq y_2 = \phi_0 + c_1\phi_1 + c_2\phi_2$ where c_2 is estimated with re-estimation of c_1. This may be followed by other approximations where in each approximation (say the i^{th} approximation) the value of the constant c_i is estimated while the values of $c_1, \cdots, c_{i-1}$ are re-estimated. The main issue (and the fundamental presumption) in this gradual process is that each approximation is better than (or at least not inferior to) the previous approximation so that we are always heading toward better approximations to the real solution hoping that in the end (i.e. if we continue with this process) we get very close to the solution y (or we may even get to the solution itself if we are lucky and made good choices). In fact, if this procedure develops as described above (i.e. the approximations improve persistently) then we should expect that we can make our approximation as close as we wish to the real solution by increasing the order of approximation (as represented by n) and hence we should expect to converge to the real solution when $n \to \infty$, that is:[139]

$$y = \phi_0 + \sum_{i=1}^{\infty} c_i\phi_i \tag{135}$$

(**f**) The above-described Rayleigh-Ritz method is one dimensional. The method can be easily generalized to multi-dimensions (although the required algebra and mathematical manipulation become very lengthy and messy). However, instead of going through the description of this simple generalization we will demonstrate this generalization by some examples of the Rayleigh-Ritz method in 2D (see Problems 6-9).

Problems

1. Describe the Rayleigh-Ritz method in a few words.
   Answer: It is a variational method that employs basis functions to find approximate solutions for boundary-value variational problems.
2. Re-solve part (a) of Problem 12 of § 1.4 (with $k = 1$) using this time the Rayleigh-Ritz method in a gradual process. Plot the obtained approximation in the end of each stage of approximation (starting from y_1) and hence stop this gradual process when the obtained solution is sufficiently close (visually) to the analytical (exact) solution that you obtained in Problem 12 of § 1.4.
   Answer: Noting that $k = 1$, we have $I[y] = \int_{x_1}^{x_2} \left(y'^2 + y^2\right) dx$ with $y\,(x_1 = 0) = 0$ and $y\,(x_2 = 1) = 1$. We assume that we have no idea about the general form of the real solution and hence we use polynomial basis functions. Referring to point (d) in the text, we select the zeroth basis function ϕ_0 to satisfy the given boundary conditions and select all the other basis functions ϕ_i ($i = 1, \cdots, n$) to vanish at the boundaries. So, if we choose $\phi_0 = x$ (which is the straight line passing through the two boundary points) then the boundary conditions are satisfied by this function because $\phi_0(0) = 0$ and $\phi_0(1) = 1$. Also, if we choose the other basis functions so that they all contain the factor $x(x-1)$ then they will all vanish at the boundaries because the x factor will ensure the vanishing at $x = 0$ while the $(x-1)$ factor will ensure the vanishing at $x = 1$. Accordingly, we can write the n^{th} approximation y_n (i.e. the approximation obtained in the n^{th} stage of the gradual process) as:

$$y_n = x + x\,(x-1)\left[c_1 + c_2 x + \cdots + c_n x^{n-1}\right]$$

---

[138] It may be more appropriate to say: while the other basis functions are chosen so that all the terms involving these functions vanish at the boundaries. This depends on the meaning of "basis functions". Anyway, this is a trivial matter.

[139] In fact, there are other conditions and restrictions and hence the above description is not sufficiently rigorous.

As we see, this approximation obviously satisfies the two boundary conditions at any stage of this process.
Now, the first approximation is:

$$y_1 = x + c_1 x\,(x-1) = c_1 x^2 + (1-c_1)\,x \tag{136}$$

On substituting this into the integrand of the functional integral we get:

$$\begin{aligned} y'^2 + y^2 &\simeq [2c_1 x + (1-c_1)]^2 + \left[c_1 x^2 + (1-c_1)\,x\right]^2 \\ &= 4c_1^2 x^2 + 4c_1(1-c_1)\,x + (1-c_1)^2 + c_1^2 x^4 + 2c_1(1-c_1)\,x^3 + (1-c_1)^2 x^2 \\ &= 4c_1^2x^2 + 4c_1x - 4c_1^2x + 1 - 2c_1 + c_1^2 + c_1^2x^4 + 2c_1x^3 - 2c_1^2x^3 + x^2 - 2c_1x^2 + c_1^2x^2 \\ &= c_1^2x^4 + \left(2c_1 - 2c_1^2\right)x^3 + \left(5c_1^2 - 2c_1 + 1\right)x^2 + \left(4c_1 - 4c_1^2\right)x + \left(c_1^2 - 2c_1 + 1\right) \end{aligned}$$

On substituting this into the functional integral and integrating we get:

$$\begin{aligned} I &\simeq \left[\frac{c_1^2}{5}x^5 + \left(\frac{2c_1 - 2c_1^2}{4}\right)x^4 + \left(\frac{5c_1^2 - 2c_1 + 1}{3}\right)x^3 + \left(\frac{4c_1 - 4c_1^2}{2}\right)x^2 + \left(c_1^2 - 2c_1 + 1\right)x\right]_0^1 \\ &= \left[\frac{c_1^2}{5} + \left(\frac{2c_1 - 2c_1^2}{4}\right) + \left(\frac{5c_1^2 - 2c_1 + 1}{3}\right) + \left(\frac{4c_1 - 4c_1^2}{2}\right) + \left(c_1^2 - 2c_1 + 1\right)\right] - 0 \\ &= \frac{11}{30}c_1^2 - \frac{1}{6}c_1 + \frac{4}{3} \end{aligned}$$

On differentiating I with respect to c_1 and setting the result to zero we get:

$$\frac{dI}{dc_1} = \frac{11}{15}c_1 - \frac{1}{6} = 0 \qquad \text{and hence} \qquad c_1 = \frac{5}{22}$$

So, the first approximation is (see Eq. 136):

$$y_1 = \frac{5}{22}x^2 + \left(1 - \frac{5}{22}\right)x = \frac{5}{22}x^2 + \frac{17}{22}x$$

On plotting this solution (plotted as circles in Figure 69) alongside the analytical solution (plotted as solid curve in Figure 69) we see that it is sufficiently close. So, we stop this gradual process.

3. Re-solve part (g) of Problem 12 of § 1.4 using this time the Rayleigh-Ritz method in a gradual process. Plot the obtained approximation in the end of each stage of approximation (starting from y_1) and hence stop this gradual process when the obtained solution is practically indistinguishable (by vision) from the analytical solution that you obtained in Problem 12 of § 1.4.
   Answer: We have $I[y] = \int_{x_1}^{x_2} \left(y'^2 - y^2 - 2xy\right) dx$ with $y\,(x_1 = 0) = 1$ and $y\,(x_2 = 1) = 2$. We assume that we have no idea about the general form of the real solution and hence we use polynomial basis functions. Referring to point (d) in the text, we select the zeroth basis function ϕ_0 to satisfy the given boundary conditions and select all the other basis functions ϕ_i $(i = 1, \cdots, n)$ to vanish at the boundaries. So, if we choose $\phi_0 = 1 + x$ (which is the straight line passing through the two boundary points) then the boundary conditions are satisfied by this function because $\phi_0(0) = 1 + 0 = 1$ and $\phi_0(1) = 1 + 1 = 2$. Also, if we choose the other basis functions so that they all contain the factor $x(x-1)$ then they will all vanish at the boundaries because the x factor will ensure the vanishing at $x = 0$ while the $(x-1)$ factor will ensure the vanishing at $x = 1$. Accordingly, we can write the n^{th} approximation y_n (i.e. the approximation obtained in the n^{th} stage of the gradual process) as:

$$y_n = 1 + x + x\,(x-1)\left[c_1 + c_2 x + \cdots + c_n x^{n-1}\right]$$

As we see, this approximation obviously satisfies the two boundary conditions at any stage of this process.
Now, the first approximation is:

$$y_1 = 1 + x + c_1 x\,(x-1) = c_1 x^2 + (1-c_1)\,x + 1 \tag{137}$$

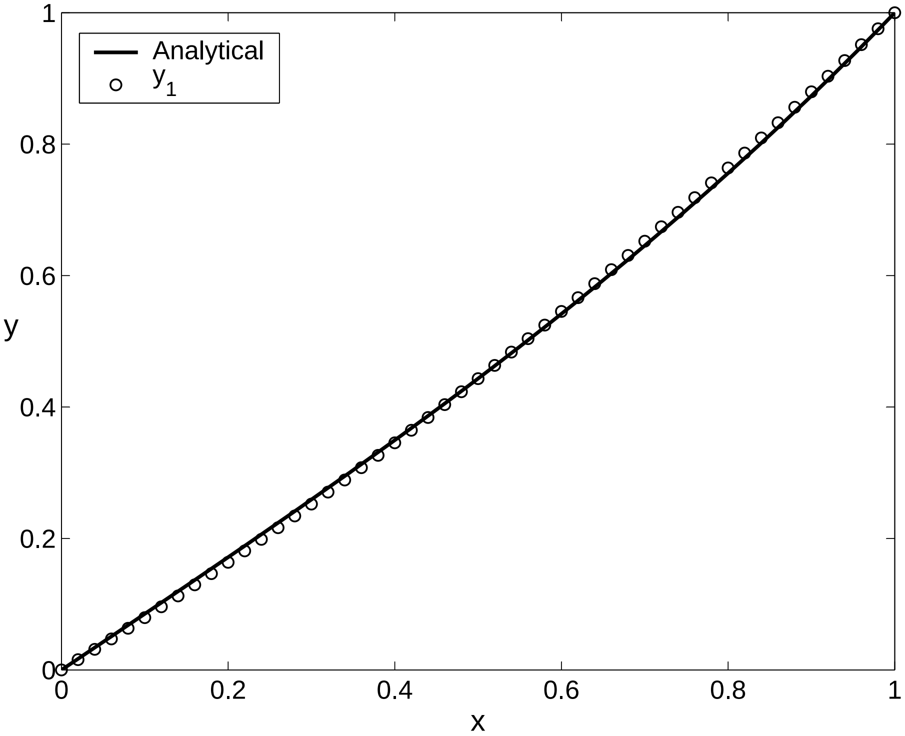

Figure 69: Plot of the analytical solution $y = \frac{\sinh(x)}{\sinh(1)}$ of Problem 2 of § 8 (as obtained in part a of Problem 12 of § 1.4 with $k = 1$) alongside the first Rayleigh-Ritz approximation $y_1 = \frac{5}{22}x^2 + \frac{17}{22}x$.

On substituting this into the integrand of the functional integral we get:

$$\begin{aligned}
y'^2 - y^2 - 2xy &\simeq [2c_1x + (1 - c_1)]^2 - \left[c_1x^2 + (1 - c_1)\,x + 1\right]^2 - 2x\left[c_1x^2 + (1 - c_1)\,x + 1\right] \\
&= \left[4c_1^2x^2 + 4c_1x - 4c_1^2x + 1 - 2c_1 + c_1^2\right] - \\
&\qquad \left[c_1^2x^4 + 2c_1x^3 - 2c_1^2x^3 + 2c_1x^2 + x^2 - 2c_1x^2 + c_1^2x^2 + 2x - 2c_1x + 1\right] - \\
&\qquad \left[2c_1x^3 + 2x^2 - 2c_1x^2 + 2x\right] \\
&= -c_1^2x^4 + \left(2c_1^2 - 4c_1\right)x^3 + \left(3c_1^2 + 2c_1 - 3\right)x^2 + \left(-4c_1^2 + 6c_1 - 4\right)x + \left(c_1^2 - 2c_1\right)
\end{aligned}$$

On substituting this into the functional integral and integrating we get:

$$\begin{aligned}
I &\simeq \left[-\frac{c_1^2}{5}x^5 + \left(\frac{2c_1^2 - 4c_1}{4}\right)x^4 + \left(\frac{3c_1^2 + 2c_1 - 3}{3}\right)x^3 + \left(\frac{-4c_1^2 + 6c_1 - 4}{2}\right)x^2 + \left(c_1^2 - 2c_1\right)x\right]_0^1 \\
&= \left[-\frac{c_1^2}{5} + \left(\frac{2c_1^2 - 4c_1}{4}\right) + \left(\frac{3c_1^2 + 2c_1 - 3}{3}\right) + \left(\frac{-4c_1^2 + 6c_1 - 4}{2}\right) + \left(c_1^2 - 2c_1\right)\right] - 0 \\
&= \frac{3}{10}c_1^2 + \frac{2}{3}c_1 - 3
\end{aligned}$$

On differentiating I with respect to c_1 and setting the result to zero we get:

$$\frac{dI}{dc_1} = \frac{6c_1}{10} + \frac{2}{3} = 0 \qquad \text{and hence} \qquad c_1 = -\frac{10}{9}$$

So, the first approximation is (see Eq. 137):

$$y_1 = -\frac{10}{9}x^2 + \frac{19}{9}x + 1$$

On plotting this solution (plotted as dashed curve in Figure 70) alongside the analytical solution (plotted as solid curve in Figure 70) we see that it is close but distinguishable. So, we go to the next approximation.

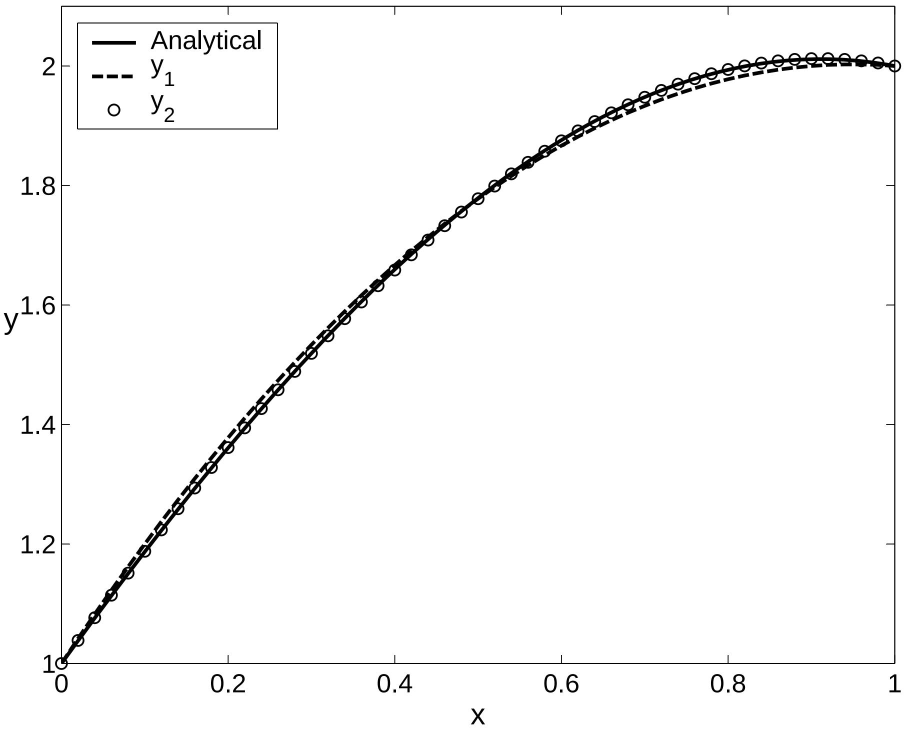

Figure 70: Plot of the analytical solution $y = \cos x + \left(\frac{3-\cos 1}{\sin 1}\right)\sin x - x$ of Problem 3 of § 8 (as obtained in part g of Problem 12 of § 1.4) alongside the first Rayleigh-Ritz approximation $y_1 = -\frac{10}{9}x^2 + \frac{19}{9}x + 1$ and the second Rayleigh-Ritz approximation $y_2 = -\frac{14}{41}x^3 - \frac{221}{369}x^2 + \frac{716}{369}x + 1$.

Now, the second approximation is:

$$y_2 = 1 + x + c_1 x(x-1) + c_2 x^2 (x-1) = c_2 x^3 + (c_1 - c_2)x^2 + (1 - c_1)x + 1 \tag{138}$$

On substituting this into the integrand of the functional integral we get:

$$\begin{aligned} y'^2 - y^2 - 2xy &\simeq \left[3c_2x^2 + 2(c_1-c_2)x + (1-c_1)\right]^2 - \left[c_2x^3 + (c_1-c_2)x^2 + (1-c_1)x + 1\right]^2 - \\ &\quad 2x\left[c_2x^3 + (c_1-c_2)x^2 + (1-c_1)x + 1\right] \\ &= -c_2^2x^6 + \left(2c_2^2 - 2c_1c_2\right)x^5 + \left(8c_2^2 - c_1^2 + 4c_1c_2 - 4c_2\right)x^4 + \\ &\quad \left(2c_1^2 + 10c_1c_2 - 12c_2^2 - 4c_1 + 2c_2\right)x^3 + \left(4c_2^2 - 14c_1c_2 + 2c_1 + 8c_2 - 3 + 3c_1^2\right)x^2 + \\ &\quad \left(4c_1c_2 - 4c_1^2 + 6c_1 - 4c_2 - 4\right)x + \left(c_1^2 - 2c_1\right) \end{aligned}$$

On substituting this into the functional integral and integrating we get:

$$\begin{aligned} I \simeq &\left[-\frac{c_2^2}{7}x^7 + \left(\frac{2c_2^2 - 2c_1c_2}{6}\right)x^6 + \left(\frac{8c_2^2 - c_1^2 + 4c_1c_2 - 4c_2}{5}\right)x^5 + \right. \\ &\left(\frac{2c_1^2 + 10c_1c_2 - 12c_2^2 - 4c_1 + 2c_2}{4}\right)x^4 + \left(\frac{4c_2^2 - 14c_1c_2 + 2c_1 + 8c_2 - 3 + 3c_1^2}{3}\right)x^3 + \end{aligned}$$

$$
\begin{aligned}
&\left(\frac{4c_1c_2-4c_1^2+6c_1-4c_2-4}{2}\right)x^2+\left(c_1^2-2c_1\right)x\Bigg]_0^1\\
=\ &\Bigg[-\frac{c_2^2}{7}+\left(\frac{2c_2^2-2c_1c_2}{6}\right)+\left(\frac{8c_2^2-c_1^2+4c_1c_2-4c_2}{5}\right)+\\
&\left(\frac{2c_1^2+10c_1c_2-12c_2^2-4c_1+2c_2}{4}\right)+\left(\frac{4c_2^2-14c_1c_2+2c_1+8c_2-3+3c_1^2}{3}\right)+\\
&\left(\frac{4c_1c_2-4c_1^2+6c_1-4c_2-4}{2}\right)+\left(c_1^2-2c_1\right)\Bigg]-0\\
=\ &\frac{3}{10}c_1^2+\frac{3}{10}c_1c_2+\frac{13}{105}c_2^2+\frac{2}{3}c_1+\frac{11}{30}c_2-3
\end{aligned}
$$

On differentiating I with respect to c_1 and c_2 and setting the results to zero we get:

$$
\begin{aligned}
\frac{\partial I}{\partial c_1} &= \frac{3}{5}c_1+\frac{3}{10}c_2+\frac{2}{3}=0\\
\frac{\partial I}{\partial c_2} &= \frac{3}{10}c_1+\frac{26}{105}c_2+\frac{11}{30}=0
\end{aligned}
$$

On solving this system of simultaneous equations we get $c_1=-\frac{347}{369}$ and $c_2=-\frac{14}{41}$. So, the second approximation is (see Eq. 138):

$$y_2=-\frac{14}{41}x^3+\left(-\frac{347}{369}+\frac{14}{41}\right)x^2+\left(1+\frac{347}{369}\right)x+1=-\frac{14}{41}x^3-\frac{221}{369}x^2+\frac{716}{369}x+1$$

On plotting this solution (plotted as circles in Figure 70) alongside the analytical solution (plotted as solid curve in Figure 70) we see that it is indistinguishable from the analytical solution. So, we stop this gradual process.

4. Re-solve part (c) of Problem 12 of § 1.4 using this time the Rayleigh-Ritz method in a gradual process. Plot the obtained approximation in the end of each stage of approximation (starting from y_1) and hence stop this gradual process when the obtained solution is practically indistinguishable (by vision) from the analytical solution that you obtained in Problem 12 of § 1.4.
   Answer: We have $I[y]=\int_{x_1}^{x_2}\frac{y'^2}{x^3}\,dx$ with $y\left(x_1=2\right)=1$ and $y\left(x_2=4\right)=31$. We again use polynomial basis functions. Referring to point (d) in the text, we select the zeroth basis function ϕ_0 to satisfy the given boundary conditions and select all the other basis functions ϕ_i $(i=1,\cdots,n)$ to vanish at the boundaries. So, if we choose $\phi_0=15x-29$ (which is the straight line passing through the two boundary points) then the boundary conditions are satisfied by this function because $\phi_0(2)=30-29=1$ and $\phi_0(4)=60-29=31$. Also, if we choose the other basis functions so that they all contain the factor $(x-2)(x-4)$ then they will all vanish at the boundaries because the $(x-2)$ factor will ensure the vanishing at $x=2$ while the $(x-4)$ factor will ensure the vanishing at $x=4$. Accordingly, we can write the n^{th} approximation y_n as:

$$
\begin{aligned}
y_n &= 15x-29+(x-2)(x-4)\left[c_1+c_2x+\cdots+c_nx^{n-1}\right]\\
&= 15x-29+\left(x^2-6x+8\right)\left[c_1+c_2x+\cdots+c_nx^{n-1}\right]
\end{aligned}
$$

Now, the first approximation is:

$$y_1=15x-29+c_1(x^2-6x+8)\tag{139}$$

On substituting this into the integrand of the functional integral we get:

$$\frac{y'^2}{x^3}\simeq\frac{\left[15+c_1\left(2x-6\right)\right]^2}{x^3}$$

$$
\begin{aligned}
&= \frac{[2c_1x - 6c_1 + 15]^2}{x^3} \\
&= \frac{4c_1^2x^2 + \left(60c_1 - 24c_1^2\right)x + 36c_1^2 - 180c_1 + 225}{x^3} \\
&= 4c_1^2x^{-1} + \left(60c_1 - 24c_1^2\right)x^{-2} + \left(36c_1^2 - 180c_1 + 225\right)x^{-3}
\end{aligned}
$$

On substituting this into the functional integral and integrating we get:

$$
\begin{aligned}
I &\simeq \left[4c_1^2 \ln x + \left(24c_1^2 - 60c_1\right)x^{-1} - \left(\frac{36c_1^2 - 180c_1 + 225}{2}\right)x^{-2}\right]_2^4 \\
&= \left[4c_1^2 \ln 4 + \left(24c_1^2 - 60c_1\right)4^{-1} - \left(\frac{36c_1^2 - 180c_1 + 225}{2}\right)4^{-2}\right] - \\
&\qquad \left[4c_1^2 \ln 2 + \left(24c_1^2 - 60c_1\right)2^{-1} - \left(\frac{36c_1^2 - 180c_1 + 225}{2}\right)2^{-2}\right] \\
&= \left(\frac{128 \ln 2 - 84}{32}\right)c_1^2 - \frac{15}{8}c_1 + \frac{675}{32}
\end{aligned}
$$

On differentiating I with respect to c_1 and setting the result to zero we get:

$$
\frac{dI}{dc_1} = 2\left(\frac{128 \ln 2 - 84}{32}\right)c_1 - \frac{15}{8} = 0 \qquad \text{and hence} \qquad c_1 \simeq 6.352111366
$$

So, the first approximation is (see Eq. 139):

$$
y_1 = 15x - 29 + 6.352111366\left(x^2 - 6x + 8\right)
$$

On plotting this solution (plotted as dashed curve in Figure 71) alongside the analytical solution (plotted as solid curve in Figure 71) we see that it is close but distinguishable. So, we go to the next approximation.
Now, the second approximation is:

$$
y_2 = 15x - 29 + (x^2 - 6x + 8)(c_1 + c_2x) \tag{140}
$$

On substituting this into the integrand of the functional integral we get:

$$
\begin{aligned}
\frac{y'^2}{x^3} &\simeq \frac{\left[15 + (2x - 6)(c_1 + c_2x) + c_2\left(x^2 - 6x + 8\right)\right]^2}{x^3} \\
&= 9c_2^2x + \left(12c_1c_2 - 72c_2^2\right) + \left(4c_1^2 - 84c_1c_2 + 192c_2^2 + 90c_2\right)x^{-1} + \\
&\qquad \left(176c_1c_2 - 24c_1^2 - 192c_2^2 + 60c_1 - 360c_2\right)x^{-2} + \\
&\qquad \left(36c_1^2 - 96c_1c_2 + 64c_2^2 - 180c_1 + 240c_2 + 225\right)x^{-3}
\end{aligned}
$$

On substituting this into the functional integral and integrating we get:

$$
\begin{aligned}
I &\simeq \Bigg[\frac{9}{2}c_2^2x^2 + \left(12c_1c_2 - 72c_2^2\right)x + \left(4c_1^2 - 84c_1c_2 + 192c_2^2 + 90c_2\right)\ln x - \\
&\qquad \left(176c_1c_2 - 24c_1^2 - 192c_2^2 + 60c_1 - 360c_2\right)x^{-1} - \\
&\qquad \frac{1}{2}\left(36c_1^2 - 96c_1c_2 + 64c_2^2 - 180c_1 + 240c_2 + 225\right)x^{-2}\Bigg]_2^4 \\
&= \frac{1}{32}\Bigg[(128 \ln 2 - 84)c_1^2 + (1888 - 2688 \ln 2)c_1c_2 + (6144 \ln 2 - 4224)c_2^2 - 60c_1 +
\end{aligned}
$$

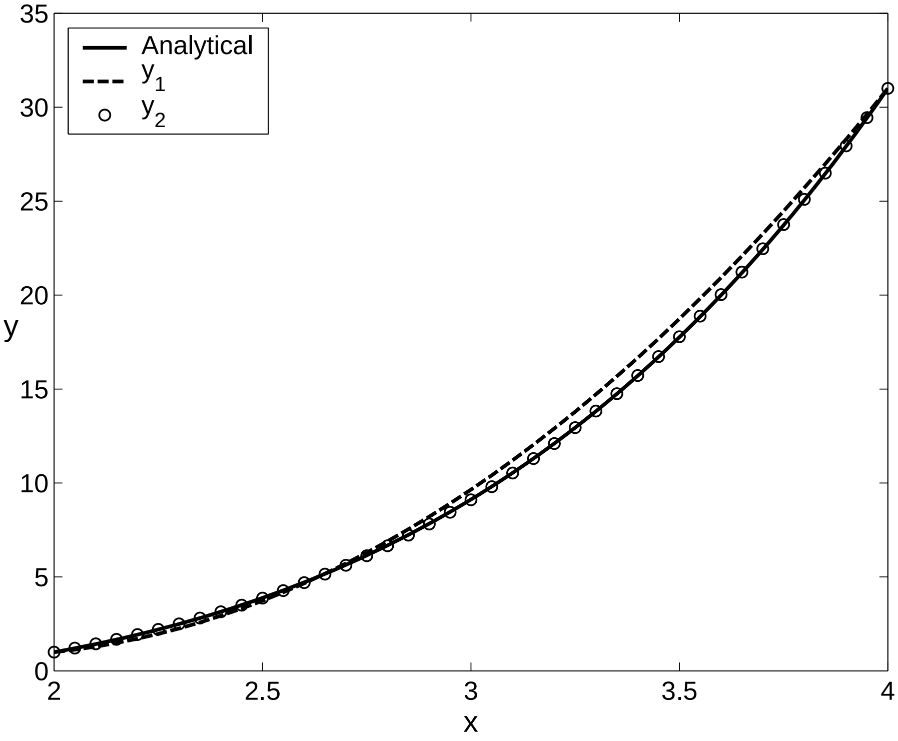

Figure 71: Plot of the analytical solution $y = \frac{x^4}{8} - 1$ of Problem 4 of § 8 (as obtained in part c of Problem 12 of § 1.4) alongside the first Rayleigh-Ritz approximation $y_1 = 15x - 29 + 6.352111366(x^2 - 6x + 8)$ and the second Rayleigh-Ritz approximation $y_2 = 15x - 29 + (x^2 - 6x + 8)(2.526375488 + 1.455927429x)$.

$$\left.(2880 \ln 2 - 2160)\, c_2 + 675 \right]$$

On differentiating I with respect to c_1 and c_2 and setting the results to zero we get:

$$\begin{aligned}
\frac{\partial I}{\partial c_1} &= 2\,(128 \ln 2 - 84)\, c_1 + (1888 - 2688 \ln 2)\, c_2 - 60 = 0 \\
\frac{\partial I}{\partial c_2} &= (1888 - 2688 \ln 2)\, c_1 + 2\,(6144 \ln 2 - 4224)\, c_2 + (2880 \ln 2 - 2160) = 0
\end{aligned}$$

On solving this system of simultaneous equations we get $c_1 \simeq 2.526375488$ and $c_2 \simeq 1.455927429$. So, the second approximation is (see Eq. 140):

$$y_2 = 15x - 29 + (x^2 - 6x + 8)(2.526375488 + 1.455927429x)$$

On plotting this solution (plotted as circles in Figure 71) alongside the analytical solution (plotted as solid curve in Figure 71) we see that it is indistinguishable from the analytical solution. So, we stop this gradual process.

5. Re-solve part (e) of Problem 12 of § 1.4 using the y_5 Rayleigh-Ritz approximation. Plot the obtained y_5 approximation alongside the analytical solution that you obtained in Problem 12 of § 1.4.
Answer: Following a similar method to that used in the previous Problems, we have:

$$\begin{aligned}
y_5 &= \frac{2}{\pi}x + x\left(x - \frac{\pi}{2}\right)\left[c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4\right] \\
&= c_5 x^6 + \left(c_4 - \frac{\pi}{2}c_5\right)x^5 + \left(c_3 - \frac{\pi}{2}c_4\right)x^4 +
\end{aligned}$$

$$\left(c_2-\frac{\pi}{2}c_3\right)x^3+\left(c_1-\frac{\pi}{2}c_2\right)x^2+\left(\frac{2}{\pi}-\frac{\pi}{2}c_1\right)x \tag{141}$$

On substituting this into the functional integral and integrating we get:

$$\begin{aligned}
I &= \int_0^{\pi/2}\left(y'^2-y^2+y\cosh x\right)\,dx \\
&\simeq \int_0^{\pi/2} dx\left[6c_5x^5+5\left(c_4-\frac{\pi}{2}c_5\right)x^4+4\left(c_3-\frac{\pi}{2}c_4\right)x^3+\right. \\
&\qquad\left.3\left(c_2-\frac{\pi}{2}c_3\right)x^2+2\left(c_1-\frac{\pi}{2}c_2\right)x+\left(\frac{2}{\pi}-\frac{\pi}{2}c_1\right)\right]^2 \\
&\quad-\left[c_5x^6+\left(c_4-\frac{\pi}{2}c_5\right)x^5+\left(c_3-\frac{\pi}{2}c_4\right)x^4+\right. \\
&\qquad\left.\left(c_2-\frac{\pi}{2}c_3\right)x^3+\left(c_1-\frac{\pi}{2}c_2\right)x^2+\left(\frac{2}{\pi}-\frac{\pi}{2}c_1\right)x\right]^2 \\
&\quad+\left[c_5x^6+\left(c_4-\frac{\pi}{2}c_5\right)x^5+\left(c_3-\frac{\pi}{2}c_4\right)x^4+\right. \\
&\qquad\left.\left(c_2-\frac{\pi}{2}c_3\right)x^3+\left(c_1-\frac{\pi}{2}c_2\right)x^2+\left(\frac{2}{\pi}-\frac{\pi}{2}c_1\right)x\right]\cosh x \\
&\simeq 0.973158c_1^2+1.528633c_1c_2+1.463175c_1c_3+1.561649c_1c_4+1.785171c_1c_5+ \\
&\quad 1.050358c_2^2+2.563096c_2c_3+3.133517c_2c_4+3.904376c_2c_5+1.791482c_3^2+ \\
&\quad 4.786866c_3c_4+6.350638c_3c_5+3.406354c_4^2+9.477671c_4c_5+6.842153c_5^2 \\
&\quad -0.263643c_1-0.177926c_2-0.157084c_3-0.161129c_4-0.181082c_5+1.453547
\end{aligned}$$

On differentiating I with respect to $c_1,\cdots,c_5$ and setting the results to zero we get:

$$\begin{aligned}
\frac{\partial I}{\partial c_1} &= 1.946316c_1+1.528633c_2+1.463175c_3+1.561649c_4+1.785171c_5-0.263643=0 \\
\frac{\partial I}{\partial c_2} &= 1.528633c_1+2.100716c_2+2.563096c_3+3.133517c_4+3.904376c_5-0.177926=0 \\
\frac{\partial I}{\partial c_3} &= 1.463175c_1+2.563096c_2+3.582964c_3+4.786866c_4+6.350638c_5-0.157084=0 \\
\frac{\partial I}{\partial c_4} &= 1.561649c_1+3.133517c_2+4.786866c_3+6.812708c_4+9.477671c_5-0.161129=0 \\
\frac{\partial I}{\partial c_5} &= 1.785171c_1+3.904376c_2+6.350638c_3+9.477671c_4+13.684306c_5-0.181082=0
\end{aligned}$$

On solving this system of simultaneous equations we get $c_1\simeq 0.1680159606$, $c_2\simeq -0.05218684043$, $c_3\simeq 0.006210311128$, $c_4\simeq 0.004267046318$ and $c_5\simeq 0.0003668967898$. So, the y_5 Rayleigh-Ritz approximation is (see Eq. 141):

$$\begin{aligned}
y_5 \simeq\ & 0.0003668967898x^6+0.003690726188269x^5-0.000492349554578x^4 \\
& -0.061941974338116x^3+0.249990857854475x^2+0.372700918614185x
\end{aligned}$$

On plotting this solution (plotted as circles in Figure 72) alongside the analytical solution (plotted as solid curve in Figure 72) we see that the y_5 Rayleigh-Ritz approximation is virtually identical to the analytical solution.

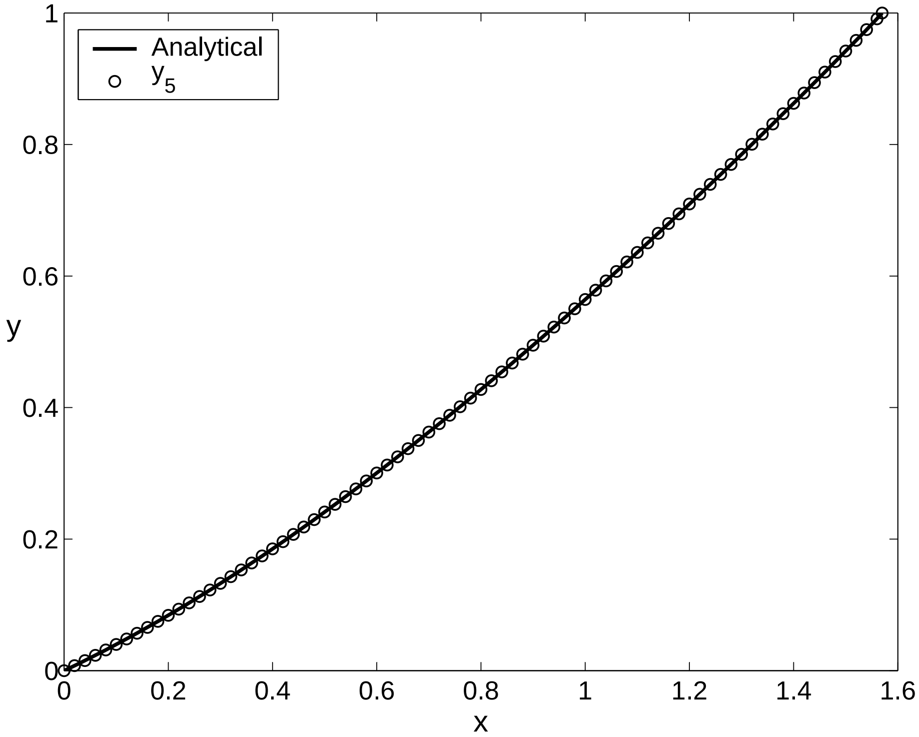

Figure 72: Plot of the analytical solution $y = -\frac{1}{4}\cos x + \left(1 - \frac{1}{4}\cosh\frac{\pi}{2}\right)\sin x + \frac{1}{4}\cosh x$ of Problem 5 of § 8 (as obtained in part e of Problem 12 of § 1.4) alongside the y_5 Rayleigh-Ritz approximation.

6. Re-solve Problem 6 of § 1.6 using this time the Rayleigh-Ritz method. Plot the obtained approximation of the Rayleigh-Ritz method and compare it to the analytical solution that you obtained in Problem 6 of § 1.6.
   Answer: This is a 2D problem and hence in the following we will extend (rather briefly) the above-described 1D Rayleigh-Ritz method to 2D. We have $I[z] = \int_0^1\int_0^1\left(z_x^2 - z_y^2\right)dx\,dy$ with $z(0,y) = z(x,0) = z(1,y) = z(x,1) = 0$ and $z(0.5,0.5) = 1$. We again use polynomial basis functions. Referring to point (d) in the text (and noting the extension to 2D), we select the zeroth basis function ϕ_0 to satisfy the given boundary conditions and select all the other basis functions ϕ_{ij} to vanish at the boundaries. So, if we choose $\phi_0 = 0$ (which is the plane $z = 0$ that passes through the four boundary lines) then the above boundary conditions are obviously satisfied by this function. Also, if we choose the other basis functions so that they all contain the factor $xy(x-1)(y-1)$ then they will all vanish at the boundaries because the factors $x, y, (x-1), (y-1)$ will ensure the vanishing at $(0,y), (x,0), (1,y), (x,1)$ respectively. Accordingly, we can write the mn^{th} approximation z_{mn} as:

$$\begin{aligned} z_{mn} &= 0 + xy(x-1)(y-1)\left[c_{11} + c_{21}x + c_{12}y + c_{22}xy + c_{31}x^2 + c_{13}y^2 + \cdots + c_{mn}x^{m-1}y^{n-1}\right] \\ &= \left(x^2y^2 - x^2y - xy^2 + xy\right)\left[c_{11} + c_{21}x + c_{12}y + c_{22}xy + c_{31}x^2 + c_{13}y^2 + \cdots + c_{mn}x^{m-1}y^{n-1}\right] \end{aligned}$$

   So, let try the z_{22} approximation, that is:

$$\begin{aligned} z_{22} &= \left(x^2y^2 - x^2y - xy^2 + xy\right)\left[c_{11} + c_{21}x + c_{12}y + c_{22}xy\right] \\ &= fx^3y^3 + (b-f)\,x^3y^2 + (c-f)\,x^2y^3 - bx^3y \\ &\qquad -cxy^3 + (a-b-c+f)\,x^2y^2 + (b-a)\,x^2y + (c-a)\,xy^2 + axy \end{aligned} \tag{142}$$

   where for the sake of simplicity and clarity we use in the second line (and subsequently) a, b, c, f to represent $c_{11}, c_{21}, c_{12}, c_{22}$.[140] On substituting this into the functional integral and integrating twice we get:

$$\begin{aligned}
I &= \int_0^1 \int_0^1 \left(z_x^2 - z_y^2\right) dx\, dy \\
&\simeq \int_0^1 dy \int_0^1 dx \Big[3fx^2y^3 + 3\left(b-f\right)x^2y^2 + 2\left(c-f\right)xy^3 - 3bx^2y - cy^3 + 2\left(a-b-c+f\right)xy^2 \\
&\qquad +2\left(b-a\right)xy + \left(c-a\right)y^2 + ay\Big]^2 - \Big[3fx^3y^2 + 2\left(b-f\right)x^3y + 3\left(c-f\right)x^2y^2 \\
&\qquad -bx^3 - 3cxy^2 + 2\left(a-b-c+f\right)x^2y + \left(b-a\right)x^2 + 2\left(c-a\right)xy + ax\Big]^2 \\
&= \frac{1}{210}\int_0^1 dy \Big[\left\{28f^2 + 70cf + 70c^2\right\}y^6 \\
&\qquad + \left\{-56f^2 + \left(56b - 140c + 70a\right)f - 140c^2 + \left(70b + 140a\right)c\right\}y^5 \\
&\qquad + \left\{10f^2 + \left(7c - 112b - 140a\right)f + 7c^2 - \left(140b + 280a\right)c + 28b^2 + 70ab + 70a^2\right\}y^4 \\
&\qquad + \left\{24f^2 + \left(84c + 32b + 28a\right)f + 84c^2 + \left(28b + 56a\right)c - 56b^2 - 140ab - 140a^2\right\}y^3 \\
&\qquad + \left\{-8f^2 + \left(28b - 28c + 49a\right)f - 28c^2 + \left(49b + 98a\right)c + 20b^2 + 42ab + 42a^2\right\}y^2 \\
&\qquad + \left\{-\left(8b + 14a\right)f - \left(14b + 28a\right)c + 8b^2 + 28ab + 28a^2\right\}y - \left\{2b^2 + 7ab + 7a^2\right\}\Big] \\
&= \frac{2\left(b^2 - c^2 + bf - cf\right)}{1575}
\end{aligned}$$

As we see, I is independent of a and hence a (which stands for c_{11}) is an arbitrary constant that can be set to zero, i.e. $a = 0$.[141] On differentiating I with respect to b, c, f and setting the results to zero we get:

$$\begin{aligned}
\frac{\partial I}{\partial b} &= 2b + f = 0 \\
\frac{\partial I}{\partial c} &= -2c - f = 0 \\
\frac{\partial I}{\partial f} &= b - c = 0
\end{aligned}$$

On solving this system of simultaneous equations we get $b = c = -\frac{f}{2}$. So, the z_{22} approximation is (see Eq. 142 noting that $a = 0$):

$$z_{22} = f\left(x^3y^3 - \frac{3}{2}x^3y^2 - \frac{3}{2}x^2y^3 + \frac{1}{2}x^3y + \frac{1}{2}xy^3 + 2x^2y^2 - \frac{1}{2}x^2y - \frac{1}{2}xy^2\right)$$

To determine f we use the constraint $z\left(0.5, 0.5\right) = 1$, that is:

$$f = \frac{z_{22}}{x^3y^3 - \frac{3}{2}x^3y^2 - \frac{3}{2}x^2y^3 + \frac{1}{2}x^3y + \frac{1}{2}xy^3 + 2x^2y^2 - \frac{1}{2}x^2y - \frac{1}{2}xy^2} = \frac{1}{-0.015625} = -64$$

Hence:

$$z_{22} = -64\left(x^3y^3 - \frac{3}{2}x^3y^2 - \frac{3}{2}x^2y^3 + \frac{1}{2}x^3y + \frac{1}{2}xy^3 + 2x^2y^2 - \frac{1}{2}x^2y - \frac{1}{2}xy^2\right)$$

---

[140] We use f instead of d or e to avoid potential confusion with dx and the number $e \simeq 2.71828$.

[141] In fact, this should have been anticipated earlier from the fact that $\phi_0 = 0$ and all the boundary conditions are zero and hence c_{11} (which represents the constant term) should be zero. However, we preferred to do it the long way to demonstrate this fact.

On plotting this solution (see the upper frame of Figure 73) alongside the analytical solution (see the lower frame of Figure 73) which we obtained in Problem 6 of § 1.6 we see that although the z_{22} approximation is not sufficiently accurate, it is still useful. In fact, we may expect a better result from higher order approximations although the algebra becomes increasingly messy and difficult. However, the algebraic difficulties can be overcome with automation where computer codes can take care of the required hard work.

7. Re-solve Problem 7 of § 1.6 using this time the Rayleigh-Ritz method. Plot the obtained approximation of the Rayleigh-Ritz method and compare it to the analytical solution that you obtained in Problem 7 of § 1.6.

Answer: We have $I[z] = \int_0^1 \int_0^1 \left(z_x^2 + z_y^2\right) dx\, dy$ with $z(0,y) = z(x,0) = z(1,y) = 0$ and $z(x,1) = \sin(\pi x)\sinh(\pi)$. We again use polynomial basis functions (apart from the zeroth basis function which should satisfy the boundary conditions and hence it can take any necessary form, as will be clarified next). Referring to point (d) in the text (and noting the extension to 2D), we select the zeroth basis function ϕ_0 to satisfy the given boundary conditions and select all the other basis functions ϕ_{ij} to vanish at the boundaries. So, if we choose $\phi_0 = \sin(\pi x)\sinh(\pi y)$ then the above boundary conditions are obviously satisfied by this function. Also, if we choose the other basis functions so that they all contain the factor $xy(x-1)(y-1)$ then they will all vanish at the boundaries because the factors $x, y, (x-1), (y-1)$ will ensure the vanishing at $(0,y), (x,0), (1,y), (x,1)$ respectively. Accordingly, we can write the mn^{th} approximation z_{mn} as:

$$\begin{aligned} z_{mn} &= \sin(\pi x)\sinh(\pi y) + xy(x-1)(y-1)\Big[c_{11} + c_{21}x + c_{12}y + c_{22}xy + \\ &\qquad c_{31}x^2 + c_{13}y^2 + \cdots + c_{mn}x^{m-1}y^{n-1}\Big] \\ &= \sin(\pi x)\sinh(\pi y) + \left(x^2y^2 - x^2y - xy^2 + xy\right)\Big[c_{11} + c_{21}x + c_{12}y + c_{22}xy + \\ &\qquad c_{31}x^2 + c_{13}y^2 + \cdots + c_{mn}x^{m-1}y^{n-1}\Big] \end{aligned}$$

So, let try the z_{22} approximation, that is:

$$\begin{aligned} z_{22} &= \sin(\pi x)\sinh(\pi y) + \left(x^2y^2 - x^2y - xy^2 + xy\right)\left[c_{11} + c_{21}x + c_{12}y + c_{22}xy\right] \\ &= \sin(\pi x)\sinh(\pi y) + fx^3y^3 + (b-f)\,x^3y^2 + (c-f)\,x^2y^3 - bx^3y \\ &\qquad -cxy^3 + (a-b-c+f)\,x^2y^2 + (b-a)\,x^2y + (c-a)\,xy^2 + axy \end{aligned} \tag{143}$$

where for the sake of simplicity and clarity we use in the second line (and subsequently) a, b, c, f to represent $c_{11}, c_{21}, c_{12}, c_{22}$. On substituting this into the functional integral and integrating twice we get:

$$\begin{aligned} I &= \int_0^1 \int_0^1 \left(z_x^2 + z_y^2\right) dx\, dy \\ &\simeq \int_0^1 dy \int_0^1 dx \Big[\pi\cos(\pi x)\sinh(\pi y) + 3fx^2y^3 + 3(b-f)\,x^2y^2 + 2(c-f)\,xy^3 \\ &\qquad -3bx^2y - cy^3 + 2(a-b-c+f)\,xy^2 + 2(b-a)\,xy + (c-a)\,y^2 + ay\Big]^2 + \\ &\qquad \Big[\pi\sin(\pi x)\cosh(\pi y) + 3fx^3y^2 + 2(b-f)\,x^3y + 3(c-f)\,x^2y^2 \\ &\qquad -bx^3 - 3cxy^2 + 2(a-b-c+f)\,x^2y + (b-a)\,x^2 + 2(c-a)\,xy + ax\Big]^2 \\ &= \frac{1}{12600}\Big[32f^2 + 4\left\{24(c+b) + 35a\right\}f + 96c^2 + 140(b+2a)\,c + 96b^2 + \\ &\qquad 35\left\{8ab + 8a^2 + 45\pi\left(e^{2\pi} - e^{-2\pi}\right)\right\}\Big] \end{aligned}$$

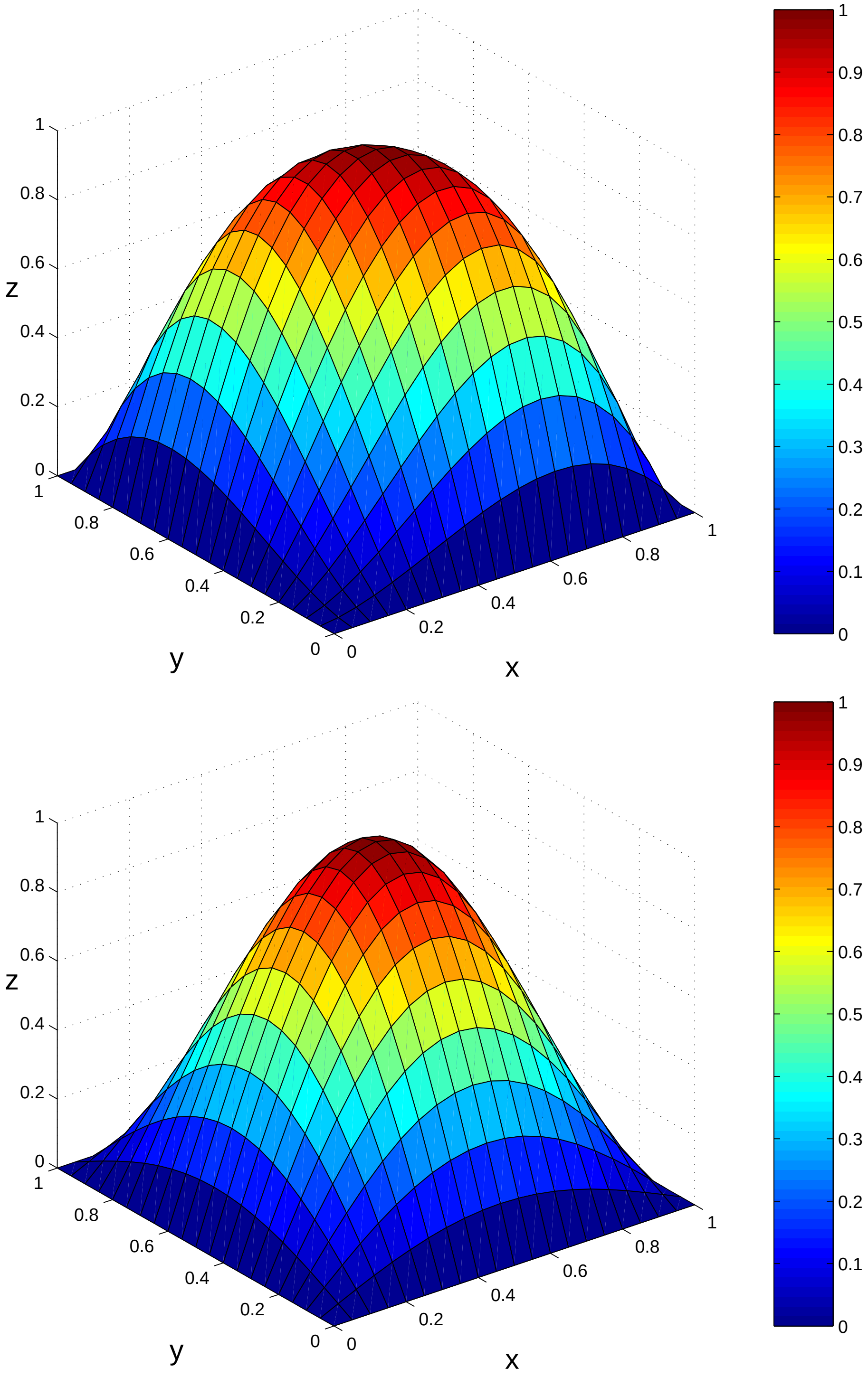

Figure 73: Plot of the z_{22} approximation of the Rayleigh-Ritz method of Problem 6 of § 8 (upper frame) alongside the analytical solution $z = \sin(\pi x)\sin(\pi y)$ of Problem 6 of § 1.6 (lower frame). For fair comparison, the same xy mesh is used in both plots.

On differentiating I with respect to a, b, c, f and setting the results to zero we get:

$$\begin{aligned}
\frac{\partial I}{\partial a} &= 140f + 280c + 280b + 560a = 0 \\
\frac{\partial I}{\partial b} &= 96f + 140c + 192b + 280a = 0 \\
\frac{\partial I}{\partial c} &= 96f + 192c + 140b + 280a = 0 \\
\frac{\partial I}{\partial f} &= 64f + 96c + 96b + 140a = 0
\end{aligned}$$

On solving this system of simultaneous equations we get $a = b = c = f = 0$. The reason is that our "approximate" function (i.e. z_{22}) contains the analytical solution (as represented by ϕ_0) and hence all the other terms should be zero. So, our z_{22} "approximation" is the exact solution that we obtained in Problem 7 of § 1.6, i.e. $z = \sin(\pi x)\sinh(\pi y)$. Hence, the plot of the z_{22} "approximation" is the same as the plot of the analytical solution that we obtained in Problem 7 of § 1.6 (see Figure 2).

8. Re-solve Problem 8 of § 1.6 using this time the Rayleigh-Ritz method. Plot the obtained approximation of the Rayleigh-Ritz method and compare it to the analytical solution that you obtained in Problem 8 of § 1.6.

Answer: We have $I[z] = \int_0^1 \int_0^1 \left(z_x^2 + z_y^2 - \pi^2 z^2 + \frac{2z^2}{y^2} \right) dx\,dy$ with $z(0,y) = z(x,0) = z(1,y) = 0$ and $z(x,1) = \sin(\pi x)$. We again use polynomial basis functions (apart from the zeroth basis function which is determined by the boundary conditions and hence it can take any necessary form, as will be clarified next). Referring to point (d) in the text (and noting the extension to 2D), we select the zeroth basis function ϕ_0 to satisfy the given boundary conditions and select all the other basis functions ϕ_{ij} to vanish at the boundaries. So, if we choose $\phi_0 = y\sin(\pi x)$ then the above boundary conditions are obviously satisfied by this function. Also, if we choose the other basis functions so that they all contain the factor $xy(x-1)(y-1)$ then they will all vanish at the boundaries because the factors $x, y, (x-1), (y-1)$ will ensure the vanishing at $(0,y), (x,0), (1,y), (x,1)$ respectively. Accordingly, we can write the mn^{th} approximation z_{mn} as:

$$\begin{aligned}
z_{mn} &= y\sin(\pi x) + xy(x-1)(y-1)\Big[c_{11} + c_{21}x + c_{12}y + c_{22}xy + \\
&\qquad c_{31}x^2 + c_{13}y^2 + \cdots + c_{mn}x^{m-1}y^{n-1}\Big] \\
&= y\sin(\pi x) + \left(x^2y^2 - x^2y - xy^2 + xy\right)\Big[c_{11} + c_{21}x + c_{12}y + c_{22}xy + \\
&\qquad c_{31}x^2 + c_{13}y^2 + \cdots + c_{mn}x^{m-1}y^{n-1}\Big]
\end{aligned}$$

So, let try the z_{22} approximation, that is:

$$\begin{aligned}
z_{22} &= y\sin(\pi x) + \left(x^2y^2 - x^2y - xy^2 + xy\right)[c_{11} + c_{21}x + c_{12}y + c_{22}xy] \\
&= y\sin(\pi x) + fx^3y^3 + (b-f)x^3y^2 + (c-f)x^2y^3 - bx^3y \\
&\quad -cxy^3 + (a-b-c+f)x^2y^2 + (b-a)x^2y + (c-a)xy^2 + axy \qquad (144)
\end{aligned}$$

where for the sake of simplicity and clarity we use in the second line (and subsequently) a, b, c, f to represent $c_{11}, c_{21}, c_{12}, c_{22}$. On substituting this into the functional integral and integrating twice we get:

$$\begin{aligned}
I &= \int_0^1 \int_0^1 \left(z_x^2 + z_y^2 - \pi^2 z^2 + \frac{2z^2}{y^2} \right) dx\,dy \\
&\simeq \int_0^1 dy \int_0^1 dx \Big[\pi y\cos(\pi x) + 3fx^2y^3 + 3(b-f)x^2y^2 + 2(c-f)xy^3 - 3bx^2y
\end{aligned}$$

$$
\begin{aligned}
&-cy^3+2\left(a-b-c+f\right)xy^2+2\left(b-a\right)xy+\left(c-a\right)y^2+ay\Big]^2+\\
&\Big[\sin\left(\pi x\right)+3fx^3y^2+2\left(b-f\right)x^3y+3\left(c-f\right)x^2y^2-bx^3-3cxy^2\\
&+2\left(a-b-c+f\right)x^2y+\left(b-a\right)x^2+2\left(c-a\right)xy+ax\Big]^2+\\
&\left(\frac{2}{y^2}-\pi^2\right)\Big[y\sin\left(\pi x\right)+fx^3y^3+\left(b-f\right)x^3y^2+\left(c-f\right)x^2y^3-bx^3y\\
&-cxy^3+\left(a-b-c+f\right)x^2y^2+\left(b-a\right)x^2y+\left(c-a\right)xy^2+axy\Big]^2\\
=\;& -\frac{1}{88200\pi^3}\Big[\left(8\pi^5-280\pi^3\right)f^2+\left(28\pi^5-868\pi^3\right)cf+\left(28\pi^5-952\pi^3\right)bf\\
&+\left(49\pi^5-1470\pi^3\right)af-117600f+\left(28\pi^5-868\pi^3\right)c^2+\left(49\pi^5-1470\pi^3\right)bc\\
&+\left(98\pi^5-2940\pi^3\right)ac-235200c+\left(28\pi^5-1232\pi^3\right)b^2+\left(98\pi^5-3920\pi^3\right)ab\\
&-352800b+\left(98\pi^5-3920\pi^3\right)a^2-705600a-132300\pi^3\Big]
\end{aligned}
$$

On differentiating I with respect to a, b, c, f and setting the results to zero we get:

$$
\begin{aligned}
\frac{\partial I}{\partial a} &= 2\left(98\pi^5-3920\pi^3\right)a+\left(98\pi^5-3920\pi^3\right)b+\left(98\pi^5-2940\pi^3\right)c+\\
&\qquad\left(49\pi^5-1470\pi^3\right)f-705600=0\\
\frac{\partial I}{\partial b} &= \left(98\pi^5-3920\pi^3\right)a+2\left(28\pi^5-1232\pi^3\right)b+\left(49\pi^5-1470\pi^3\right)c+\\
&\qquad\left(28\pi^5-952\pi^3\right)f-352800=0\\
\frac{\partial I}{\partial c} &= \left(98\pi^5-2940\pi^3\right)a+\left(49\pi^5-1470\pi^3\right)b+2\left(28\pi^5-868\pi^3\right)c+\\
&\qquad\left(28\pi^5-868\pi^3\right)f-235200=0\\
\frac{\partial I}{\partial f} &= \left(49\pi^5-1470\pi^3\right)a+\left(28\pi^5-952\pi^3\right)b+\left(28\pi^5-868\pi^3\right)c+\\
&\qquad 2\left(8\pi^5-280\pi^3\right)f-117600=0
\end{aligned}
$$

On solving this system of simultaneous equations we get:

$$a=\frac{12000\pi^2-388800}{\pi^7-148\pi^5+3620\pi^3}\simeq-3.863893488 \qquad b=f=0 \qquad c=\frac{168000-16800\pi^2}{\pi^7-148\pi^5+3620\pi^3}\simeq 0.031307421$$

So, the z_{22} approximation is (see Eq. 144):

$$
\begin{aligned}
z_{22} \;\simeq\;& y\sin\left(\pi x\right)+0.031307421x^2y^3-0.031307421xy^3-3.895200909x^2y^2+\\
&3.863893488x^2y+3.895200909xy^2-3.863893488xy
\end{aligned}
$$

On plotting this solution (see the upper frame of Figure 74) alongside the analytical solution (see the lower frame of Figure 74) which we obtained in Problem 8 of § 1.6 we see that the z_{22} approximation is almost identical to the analytical solution. In fact, we should expect a better result from higher order approximations (e.g. z_{33}) although the algebra becomes increasingly messy and difficult.

9. Re-solve Problem 10 of § 1.6 using this time the Rayleigh-Ritz method. Plot the obtained approximation of the Rayleigh-Ritz method and compare it to the analytical (series) solution that you obtained in Problem 10 of § 1.6.
Answer: We have $I\left[z\right]=\int_0^1\int_0^1\left(z_x^2+z_y^2-4z\right)\,dx\,dy$ with $z\left(0,y\right)=z\left(x,0\right)=z\left(1,y\right)=z\left(x,1\right)=0$. In fact, the domain and boundary conditions for this Problem are the same as those of Problem 6 and

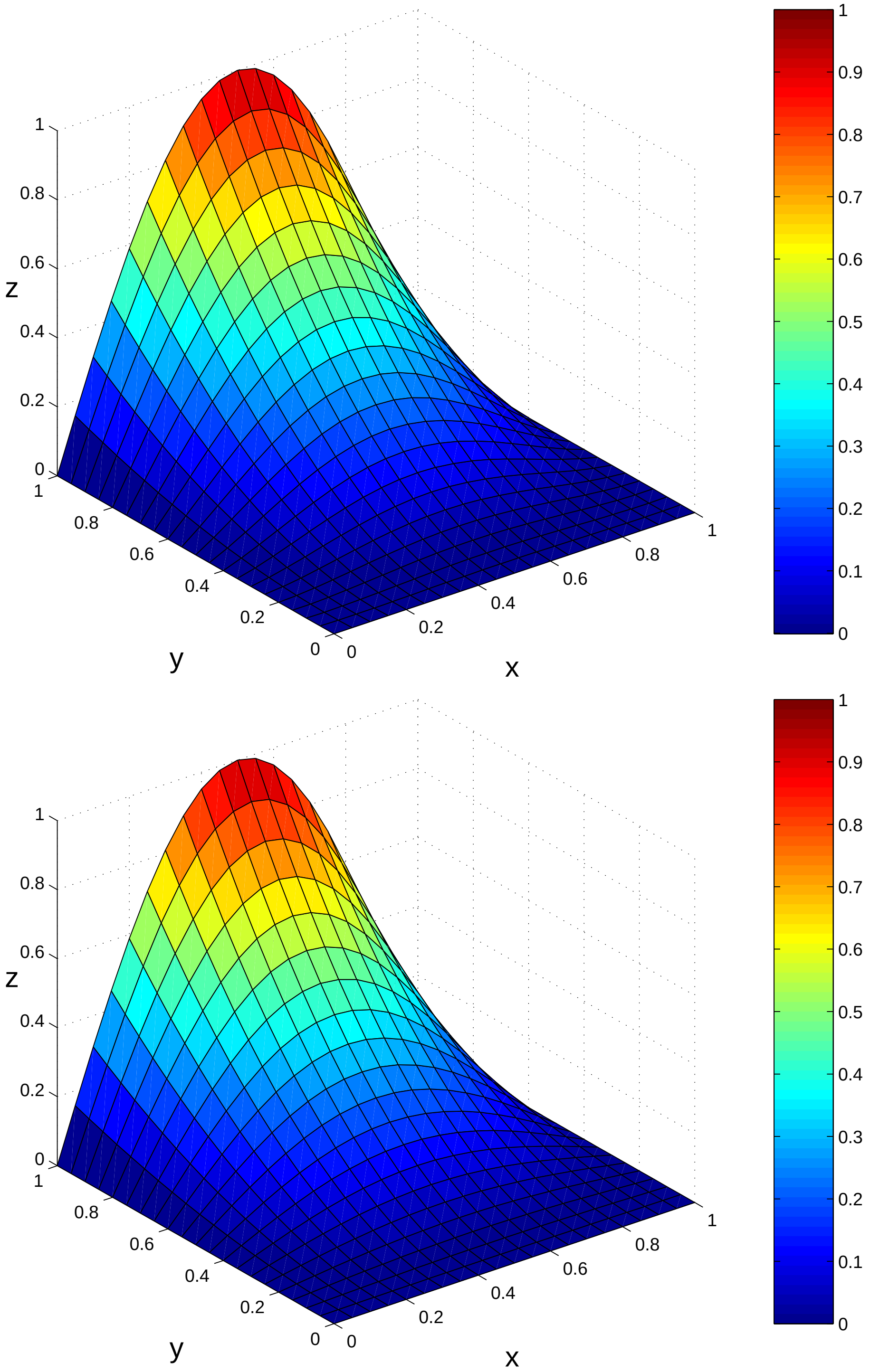

Figure 74: Plot of the z_{22} approximation of the Rayleigh-Ritz method of Problem 8 of § 8 (upper frame) alongside the analytical solution $z = y^2 \sin(\pi x)$ of Problem 8 of § 1.6 (lower frame). For fair comparison, the same xy mesh is used in both plots.

hence the basic formulation (as represented by z_{mn}) is the same. So, let try the z_{33} approximation, that is:

$$\begin{aligned}
z_{33} &= \left(x^2y^2 - x^2y - xy^2 + xy\right)\Big[c_{11} + c_{12}y + c_{13}y^2 + c_{21}x + c_{22}xy + c_{23}xy^2 \\
&\qquad + c_{31}x^2 + c_{32}x^2y + c_{33}x^2y^2\Big] \\
&= kx^4y^4 + (h-k)\,x^3y^4 + (j-k)\,x^4y^3 + (c-h)\,x^2y^4 + (i-j)\,x^4y^2 + (g-h-j+k)\,x^3y^3 + \\
&\quad (b-c-g+h)\,x^2y^3 - cxy^4 + (f-g-i+j)\,x^3y^2 - ix^4y + (a-b-f+g)\,x^2y^2 + \\
&\quad (c-b)\,xy^3 + (i-f)\,x^3y + (f-a)\,x^2y + (b-a)\,xy^2 + axy
\end{aligned} \tag{145}$$

where for the sake of simplicity we use in the second equality (and subsequently) $a, b, c, f, g, h, i, j, k$ to represent $c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, c_{33}$. On substituting this into the functional integral and integrating twice we get:

$$\begin{aligned}
I &= \int_0^1\int_0^1 \left(z_x^2 + z_y^2 - 4z\right)\,dx\,dy \\
&\simeq \int_0^1 dy\int_0^1 dx\,\Big[4kx^3y^4 + 3\,(h-k)\,x^2y^4 + 4\,(j-k)\,x^3y^3 + 2\,(c-h)\,xy^4 + 4\,(i-j)\,x^3y^2 \\
&\qquad +3\,(g-h-j+k)\,x^2y^3 + 2\,(b-c-g+h)\,xy^3 - cy^4 + 3\,(f-g-i+j)\,x^2y^2 - 4ix^3y \\
&\qquad +2\,(a-b-f+g)\,xy^2 + (c-b)\,y^3 + 3\,(i-f)\,x^2y + 2\,(f-a)\,xy + (b-a)\,y^2 + ay\Big]^2 + \\
&\quad \Big[4kx^4y^3 + 4\,(h-k)\,x^3y^3 + 3\,(j-k)\,x^4y^2 + 4\,(c-h)\,x^2y^3 + 2\,(i-j)\,x^4y \\
&\qquad +3\,(g-h-j+k)\,x^3y^2 + 3\,(b-c-g+h)\,x^2y^2 - 4cxy^3 + 2\,(f-g-i+j)\,x^3y - ix^4 \\
&\qquad +2\,(a-b-f+g)\,x^2y + 3\,(c-b)\,xy^2 + (i-f)\,x^3 + (f-a)\,x^2 + 2\,(b-a)\,xy + ax\Big]^2 - \\
&\quad 4\Big[kx^4y^4 + (h-k)\,x^3y^4 + (j-k)\,x^4y^3 + (c-h)\,x^2y^4 + (i-j)\,x^4y^2 \\
&\qquad + (g-h-j+k)\,x^3y^3 + (b-c-g+h)\,x^2y^3 - cxy^4 + (f-g-i+j)\,x^3y^2 - ix^4y \\
&\qquad + (a-b-f+g)\,x^2y^2 + (c-b)\,xy^3 + (i-f)\,x^3y + (f-a)\,x^2y + (b-a)\,xy^2 + axy\Big] \\
&= \frac{1}{264600}\Big[5880a^2 + 5880ab + 3444ac + 5880af + 2940ag + 1722ah + 3444ai + 1722aj + \\
&\quad 1008ak + 2016b^2 + 2814bc + 2940bf + 2016bg + 1407bh + 1722bi + 1176bj + 819bk + \\
&\quad 1106c^2 + 1722cf + 1407cg + 1106ch + 1008ci + 819cj + 642ck + 2016f^2 + 2016fg + \\
&\quad 1176fh + 2814fi + 1407fj + 819fk + 672g^2 + 924gh + 1407gi + 924gj + 630gk + \\
&\quad 356h^2 + 819hi + 630hj + 480hk + 1106i^2 + 1106ij + 642ik + 356j^2 + 480jk + \\
&\quad 180k^2 - 29400a - 14700b - 8820c - 14700f - 7350g - 4410h - 8820i - 4410j - 2646k\Big]
\end{aligned}$$

On differentiating I with respect to $a, b, c, f, g, h, i, j, k$ and setting the results to zero we get:

$$\begin{aligned}
\frac{\partial I}{\partial a} &= 11760a + 5880b + 3444c + 5880f + 2940g + 1722h + 3444i + 1722j + 1008k - 29400 = 0 \\
\frac{\partial I}{\partial b} &= 5880a + 4032b + 2814c + 2940f + 2016g + 1407h + 1722i + 1176j + 819k - 14700 = 0 \\
\frac{\partial I}{\partial c} &= 3444a + 2814b + 2212c + 1722f + 1407g + 1106h + 1008i + 819j + 642k - 8820 = 0 \\
\frac{\partial I}{\partial f} &= 5880a + 2940b + 1722c + 4032f + 2016g + 1176h + 2814i + 1407j + 819k - 14700 = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial I}{\partial g} &= 2940a + 2016b + 1407c + 2016f + 1344g + 924h + 1407i + 924j + 630k - 7350 = 0 \\
\frac{\partial I}{\partial h} &= 1722a + 1407b + 1106c + 1176f + 924g + 712h + 819i + 630j + 480k - 4410 = 0 \\
\frac{\partial I}{\partial i} &= 3444a + 1722b + 1008c + 2814f + 1407g + 819h + 2212i + 1106j + 642k - 8820 = 0 \\
\frac{\partial I}{\partial j} &= 1722a + 1176b + 819c + 1407f + 924g + 630h + 1106i + 712j + 480k - 4410 = 0 \\
\frac{\partial I}{\partial k} &= 1008a + 819b + 642c + 819f + 630g + 480h + 642i + 480j + 360k - 2646 = 0
\end{aligned}$$

On solving this system of simultaneous equations we get $a = 4$, $b = -21/4$, $c = 21/4$, $f = -21/4$, $g = 63/4$, $h = -63/4$, $i = 21/4$, $j = -63/4$ and $k = 63/4$. So, the z_{33} approximation is (see Eq. 145):

$$\begin{aligned}
z_{33} &= \frac{63}{4}x^4y^4 - \frac{63}{2}x^3y^4 - \frac{63}{2}x^4y^3 + 21x^2y^4 + 21x^4y^2 + 63x^3y^3 - 42x^2y^3 - 42x^3y^2 - \frac{21}{4}xy^4 \\
&\quad - \frac{21}{4}x^4y + \frac{121}{4}x^2y^2 + \frac{21}{2}xy^3 + \frac{21}{2}x^3y - \frac{37}{4}x^2y - \frac{37}{4}xy^2 + 4xy
\end{aligned}$$

On plotting this solution (see the upper frame of Figure 75) alongside the analytical (series) solution (see the lower frame of Figure 75 where we used in this plot all the series terms up to and including $m = n = 7$) which we obtained in Problem 10 of § 1.6 we see that the z_{33} Rayleigh-Ritz approximation is almost identical to the analytical (series) solution.

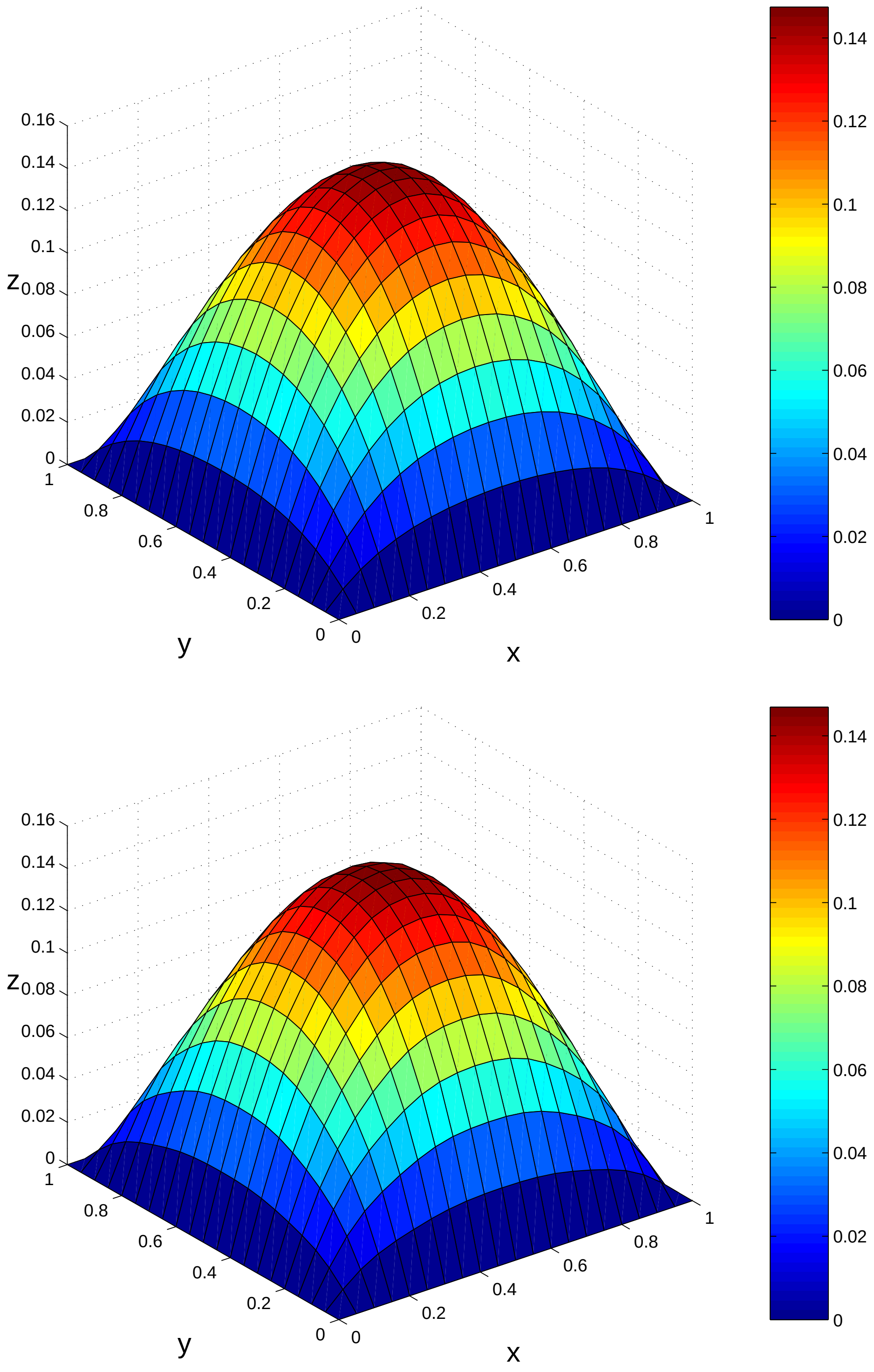

Figure 75: Plot of the z_{33} approximation of the Rayleigh-Ritz method of Problem 9 of § 8 (upper frame) alongside the analytical (series) solution $z = \frac{8}{\pi^4}\sum_{n=1}^{7}\sum_{m=1}^{7}\frac{(1-[-1]^m)(1-[-1]^n)}{mn(m^2+n^2)}\sin(m\pi x)\sin(n\pi y)$ of Problem 10 of § 1.6 (lower frame). For fair comparison, the same xy mesh is used in both plots.

Chapter 9

Numerical Methods

There are many numerical methods for solving variational problems. These methods are generally based on discretizing the continuous (analytical) formulation of the calculus of variations problems and implementing the discretized formulation version manually or computationally (using computers). In this book, we just investigate briefly one of the simplest (and possibly the simplest) numerical variational methods which is based on the finite difference technique. The idea of this method is very simple although the technique is algebraically messy in most cases (especially if it is implemented manually). This method is outlined in the following points:

1. The analytical variational formulation is fundamentally based on a functional integral (as given by Eq. 1). Now, the integral is the continuous version of the discretized version of a sum. So, we simply convert the functional integral to a sum, that is:

   $$S = \sum_{i} F\left(x_i, y_i, y_i'\right) \Delta x \tag{146}$$

   where we replaced I (for Integral) with S (for Sum) and replaced the infinitesimal dx with the finite Δx.
2. We discretize the x interval (as determined by the boundaries which represent the limits of the integral) to $n+1$ equal divisions by inserting n points evenly between the two boundaries. Accordingly, we have $\Delta x = \frac{x_{n+1} - x_0}{n+1}$ where x_0 and x_{n+1} are the x coordinates of the boundary points. So, we have $n+2$ points (i.e. 2 boundary points and n inserted points) and $n+1$ divisions. We can therefore rewrite the sum of Eq. 146 as:

   $$S = \sum_{i=0}^{n} F\left(x_i, y_i, y_i'\right) \Delta x \tag{147}$$

   where $y_i' = \frac{y_{i+1} - y_i}{\Delta x}$.
3. We can now form a table summarizing our $n+2$ points. This table shows what is known and what is unknown (which what we should look for in our solution), that is:

x_0	x_1	x_2	···············	x_{n-1}	x_n	x_{n+1}
✓	✓	✓	✓············✓	✓	✓	✓
✓	?	?	? ············ ?	?	?	✓
y_0	y_1	y_2	···············	y_{n-1}	y_n	y_{n+1}

   where the check mark ✓ means known while the question mark ? means unknown. Accordingly, S in Eq. 147 becomes a function (rather than functional) of n unknowns which are $y_1, \cdots, y_n$. We can therefore rewrite the sum of Eq. 147 as:

   $$S\left[y_1, \cdots, y_n\right] = \sum_{i=0}^{n} F\left(x_i, y_i, y_i'\right) \Delta x \tag{148}$$

   where we keep the square brackets as a reminder although S is actually a function of $y_1, \cdots, y_n$.
4. Eq. 148 means that S (which represents our "functional integral") is a function of the n variables $y_1, \cdots, y_n$ and hence to extremize S we should stationarize it with respect to these n variables simultaneously by taking the partial derivatives of S with respect to these n variables and set them to zero, that is:

   $$\frac{\partial S}{\partial y_1} = 0$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \dfrac{\partial S}{\partial y_n} & = & 0 \end{array}$$

We then solve this system of n simultaneous equations and obtain the unknowns $y_1, \cdots, y_n$.

5. So, we now have $n+2$ known points, i.e.

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2) \cdots\cdots\cdots\cdots\cdots (x_{n-1}, y_{n-1}),\ (x_n, y_n),\ (x_{n+1}, y_{n+1}) \qquad (149)$$

We can use these points as an approximation to the extremizing function either directly (and hence the extremizing function is a a polygonal curve that connects these points consecutively by straight line segments) or as an input to an interpolation schemes (such as polynomial and spline interpolations).

We should finally remark that the approximation generally improves by increasing n (although at a cost that may not be affordable or desirable at a certain point). We should also note that the above-described finite difference method is one dimensional. The method can be easily generalized to multi-dimensions (although the required algebra and mathematical manipulation become very lengthy and messy). However, instead of going through the description of this simple generalization we will demonstrate this generalization by some examples of the finite difference method in 2D (see Problems 6-8).

Problems

1. Re-solve part (c) of Problem 12 of § 1.4 using this time the finite difference method with $n = 3$ discretization scheme (i.e. by inserting 3 evenly-spaced points between the boundaries and hence making 4 divisions). Plot the obtained approximation points beside the analytical solution that you obtained in Problem 12 of § 1.4.
   Answer: From the required discretization scheme $n = 3$ and the boundary conditions $y_0\,(x_0 = 2) = 1$ and $y_4\,(x_4 = 4) = 31$ we get Δx and form the table, that is:

$$\Delta x = \frac{x_{n+1} - x_0}{n+1} = \frac{x_{3+1} - x_0}{3+1} = \frac{x_4 - x_0}{4} = \frac{4-2}{4} = \frac{1}{2}$$

x_0	x_1	x_2	x_3	x_4
2	2.5	3	3.5	4
1	?	?	?	31
y_0	y_1	y_2	y_3	y_4

Now, we use Eq. 148 with $n = 3$ and $F = y'^2/x^3$, that is:

$$\begin{aligned}
S\left[y_1, y_2, y_3\right] &= \sum_{i=0}^{3} \left[\frac{\left(y_i'\right)^2}{x_i^3}\right] \Delta x \\
&= \sum_{i=0}^{3} \left[\frac{\left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2}{x_i^3}\right] \Delta x \\
&= \sum_{i=0}^{3} \frac{\left(y_{i+1} - y_i\right)^2}{x_i^3 \Delta x} \\
&= \frac{\left(y_1 - y_0\right)^2}{x_0^3 \Delta x} + \frac{\left(y_2 - y_1\right)^2}{x_1^3 \Delta x} + \frac{\left(y_3 - y_2\right)^2}{x_2^3 \Delta x} + \frac{\left(y_4 - y_3\right)^2}{x_3^3 \Delta x} \\
&= 2\left[\frac{\left(y_1 - 1\right)^2}{8} + \frac{\left(y_2 - y_1\right)^2}{125/8} + \frac{\left(y_3 - y_2\right)^2}{27} + \frac{\left(31 - y_3\right)^2}{343/8}\right] \\
&= \frac{\left(y_1^2 - 2y_1 + 1\right)}{4} + \frac{16\left(y_2^2 - 2y_1 y_2 + y_1^2\right)}{125}
\end{aligned}$$

$$
\begin{aligned}
& + \frac{2\left(y_3^2 - 2y_2y_3 + y_2^2\right)}{27} + \frac{16\left(961 - 62y_3 + y_3^2\right)}{343} \\
= \; & \frac{189}{500}y_1^2 - \frac{32}{125}y_1y_2 + \frac{682}{3375}y_2^2 - \frac{4}{27}y_2y_3 + \frac{1118}{9261}y_3^2 - \frac{1}{2}y_1 - \frac{992}{343}y_3 + \frac{61847}{1372}
\end{aligned}
$$

On taking the partial derivatives of S with respect to y_1, y_2, y_3 and setting the results to zero we get:

$$
\begin{aligned}
\frac{\partial S}{\partial y_1} &= +\frac{189}{250}y_1 - \frac{32}{125}y_2 - \frac{1}{2} = 0 \\
\frac{\partial S}{\partial y_2} &= -\frac{32}{125}y_1 + \frac{1364}{3375}y_2 - \frac{4}{27}y_3 = 0 \\
\frac{\partial S}{\partial y_3} &= -\frac{4}{27}y_2 + \frac{2236}{9261}y_3 - \frac{992}{343} = 0
\end{aligned}
$$

On solving this system of simultaneous equations we get:

$$
y_1 = \frac{667}{187} \qquad y_2 = \frac{3209}{374} \qquad y_3 = \frac{6449}{374}
$$

On plotting the 5 points (i.e. 2 boundaries and 3 inserted) beside the analytical solution (which we obtained in Problem 12 of § 1.4) we get Figure 76.

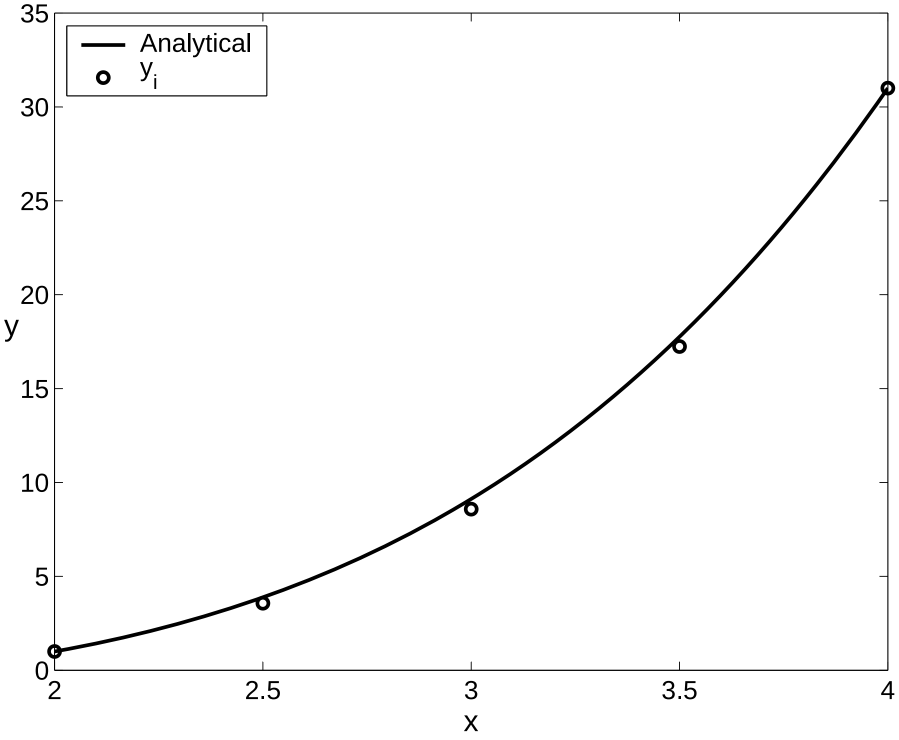

Figure 76: Plot of the analytical solution $y = \frac{x^4}{8} - 1$ of Problem 1 of § 9 (as obtained in part c of Problem 12 of § 1.4) alongside the points y_1, y_2, y_3 obtained by finite difference (as well as the boundary points).

2. Re-solve part (a) of Problem 12 of § 1.4 (with $k = 1$) using this time the finite difference method with $n = 3$ discretization scheme (i.e. by inserting 3 evenly-spaced points between the boundaries and hence making 4 divisions). Plot the obtained approximation points beside the analytical solution that you obtained in Problem 12 of § 1.4.

Answer: From the required discretization scheme $n = 3$ and the boundary conditions $y_0\left(x_0 = 0\right) = 0$ and $y_4\left(x_4 = 1\right) = 1$ we get Δx and form the table, that is:

$$\Delta x = \frac{x_{n+1} - x_0}{n+1} = \frac{x_{3+1} - x_0}{3+1} = \frac{x_4 - x_0}{4} = \frac{1-0}{4} = \frac{1}{4}$$

x_0	x_1	x_2	x_3	x_4
0	0.25	0.5	0.75	1
0	?	?	?	1
y_0	y_1	y_2	y_3	y_4

Now, we use Eq. 148 with $n = 3$ and $F = y'^2 + y^2$, that is:

$$\begin{aligned}
S\left[y_1, y_2, y_3\right] &= \sum_{i=0}^{3}\left[\left(y_i'\right)^2 + y_i^2\right]\Delta x \\
&= \sum_{i=0}^{3}\left[\left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2 + y_i^2\right]\Delta x \\
&= \sum_{i=0}^{3}\left[\frac{\left(y_{i+1} - y_i\right)^2}{\Delta x} + y_i^2\Delta x\right] \\
&= \frac{\left(y_1 - y_0\right)^2}{\Delta x} + y_0^2\Delta x + \frac{\left(y_2 - y_1\right)^2}{\Delta x} + y_1^2\Delta x + \frac{\left(y_3 - y_2\right)^2}{\Delta x} + y_2^2\Delta x + \frac{\left(y_4 - y_3\right)^2}{\Delta x} + y_3^2\Delta x \\
&= 4\left(y_1 - 0\right)^2 + \frac{0}{4} + 4\left(y_2 - y_1\right)^2 + \frac{y_1^2}{4} + 4\left(y_3 - y_2\right)^2 + \frac{y_2^2}{4} + 4\left(1 - y_3\right)^2 + \frac{y_3^2}{4} \\
&= \frac{33}{4}y_1^2 - 8y_1y_2 + \frac{33}{4}y_2^2 - 8y_2y_3 + \frac{33}{4}y_3^2 - 8y_3 + 4
\end{aligned}$$

On taking the partial derivatives of S with respect to y_1, y_2, y_3 and setting the results to zero we get:

$$\begin{aligned}
\frac{\partial S}{\partial y_1} &= +\frac{33}{2}y_1 - 8y_2 = 0 \\
\frac{\partial S}{\partial y_2} &= -8y_1 + \frac{33}{2}y_2 - 8y_3 = 0 \\
\frac{\partial S}{\partial y_3} &= -8y_2 + \frac{33}{2}y_3 - 8 = 0
\end{aligned}$$

On solving this system of simultaneous equations we get:

$$y_1 = \frac{4096}{19041} \qquad y_2 = \frac{256}{577} \qquad y_3 = \frac{13328}{19041}$$

On plotting the 5 points (i.e. 2 boundaries and 3 inserted) beside the analytical solution (which we obtained in Problem 12 of § 1.4) we get Figure 77.

3. Re-solve the previous Problem using this time $n = 4$ discretization scheme (i.e. by inserting 4 evenly-spaced points between the boundaries and hence making 5 divisions).
Answer: From the required discretization scheme $n = 4$ and the boundary conditions $y_0\left(x_0 = 0\right) = 0$ and $y_5\left(x_5 = 1\right) = 1$ we get Δx and form the table, that is:

$$\Delta x = \frac{x_{n+1} - x_0}{n+1} = \frac{x_{4+1} - x_0}{4+1} = \frac{x_5 - x_0}{5} = \frac{1-0}{5} = \frac{1}{5}$$

x_0	x_1	x_2	x_3	x_4	x_5
0	0.2	0.4	0.6	0.8	1
0	?	?	?	?	1
y_0	y_1	y_2	y_3	y_4	y_5

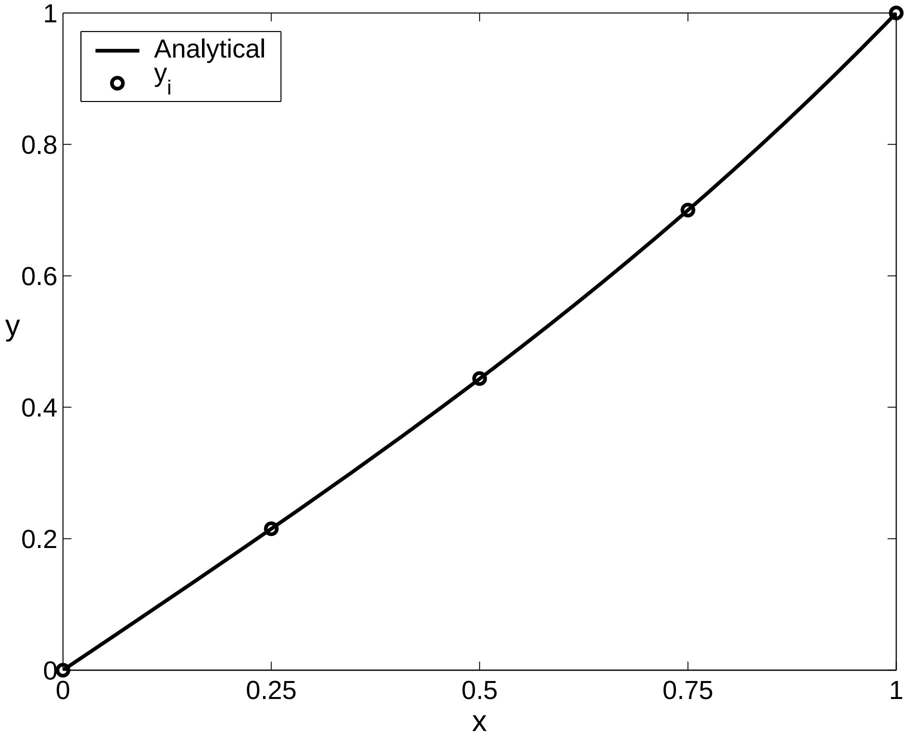

Figure 77: Plot of the analytical solution $y = \frac{\sinh(x)}{\sinh(1)}$ of Problem 2 of § 9 (as obtained in part a of Problem 12 of § 1.4 with $k = 1$) alongside the points y_1, y_2, y_3 obtained by finite difference (as well as the boundary points).

Now, we use Eq. 148 with $n = 4$ and $F = y'^2 + y^2$, that is:

$$\begin{aligned}
S\left[y_1, y_2, y_3, y_4\right] &= \sum_{i=0}^{4}\left[\left(y_i'\right)^2 + y_i^2\right]\Delta x \\
&= \sum_{i=0}^{4}\left[\left(\frac{y_{i+1}-y_i}{\Delta x}\right)^2 + y_i^2\right]\Delta x \\
&= \sum_{i=0}^{4}\left[\frac{\left(y_{i+1}-y_i\right)^2}{\Delta x} + y_i^2\Delta x\right] \\
&= \frac{\left(y_1-y_0\right)^2}{\Delta x} + y_0^2\Delta x + \frac{\left(y_2-y_1\right)^2}{\Delta x} + y_1^2\Delta x + \frac{\left(y_3-y_2\right)^2}{\Delta x} + y_2^2\Delta x \\
&\qquad + \frac{\left(y_4-y_3\right)^2}{\Delta x} + y_3^2\Delta x + \frac{\left(y_5-y_4\right)^2}{\Delta x} + y_4^2\Delta x \\
&= 5\left(y_1-0\right)^2 + \frac{0}{5} + 5\left(y_2-y_1\right)^2 + \frac{y_1^2}{5} + 5\left(y_3-y_2\right)^2 + \frac{y_2^2}{5} \\
&\qquad + 5\left(y_4-y_3\right)^2 + \frac{y_3^2}{5} + 5\left(1-y_4\right)^2 + \frac{y_4^2}{5} \\
&= \frac{51}{5}\left(y_1^2+y_2^2+y_3^2+y_4^2\right) - 10\left(y_1y_2+y_2y_3+y_3y_4+y_4\right) + 5
\end{aligned}$$

On taking the partial derivatives of S with respect to y_1, y_2, y_3, y_4 and setting the results to zero we get:

$$\begin{aligned}\frac{\partial S}{\partial y_1} &= \frac{102}{5}y_1 - 10y_2 = 0\\ \frac{\partial S}{\partial y_2} &= \frac{102}{5}y_2 - 10y_1 - 10y_3 = 0\\ \frac{\partial S}{\partial y_3} &= \frac{102}{5}y_3 - 10y_2 - 10y_4 = 0\\ \frac{\partial S}{\partial y_4} &= \frac{102}{5}y_4 - 10y_3 - 10 = 0\end{aligned}$$

On solving this system of simultaneous equations we get:

$$y_1 = \frac{390625}{2278951} \qquad y_2 = \frac{796875}{2278951} \qquad y_3 = \frac{1235000}{2278951} \qquad y_4 = \frac{1722525}{2278951}$$

On plotting the 6 points (i.e. 2 boundaries and 4 inserted) beside the analytical solution (which we obtained in Problem 12 of § 1.4) we get Figure 78.

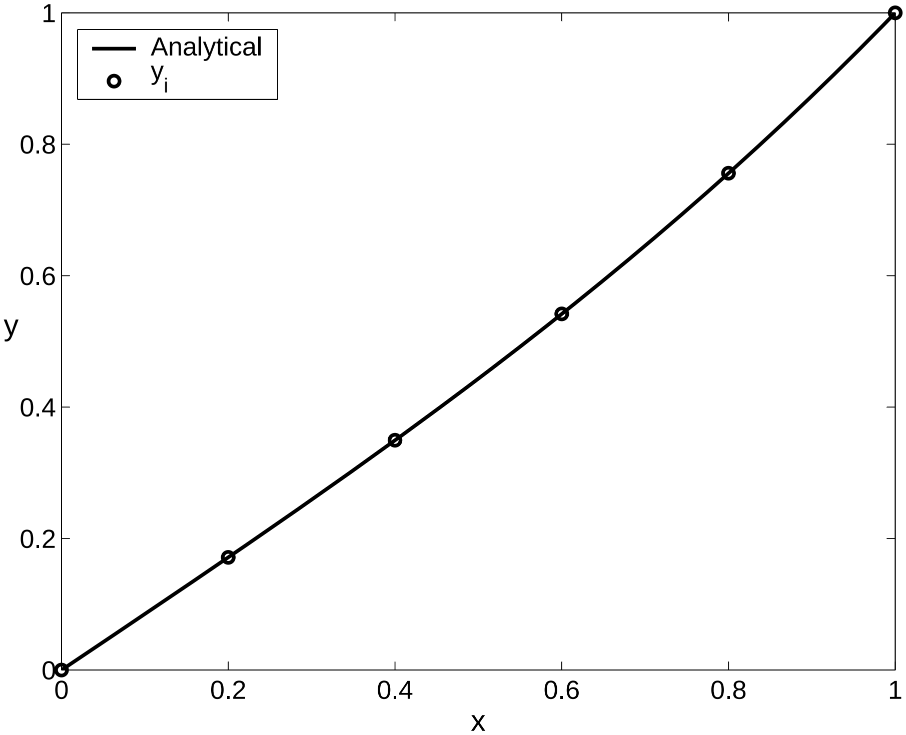

Figure 78: Plot of the analytical solution $y = \frac{\sinh(x)}{\sinh(1)}$ of Problem 3 of § 9 (as obtained in part a of Problem 12 of § 1.4 with $k = 1$) alongside the points y_1, y_2, y_3, y_4 obtained by finite difference (as well as the boundary points).

4. Re-solve part (g) of Problem 12 of § 1.4 using this time the finite difference method with $n = 4$ discretization scheme (i.e. by inserting 4 evenly-spaced points between the boundaries and hence making 5 divisions). Plot the obtained approximation points beside the analytical solution that you obtained in Problem 12 of § 1.4.

   Answer: From the required discretization scheme $n = 4$ and the boundary conditions $y_0\,(x_0 = 0) = 1$

and $y_5\,(x_5 = 1) = 2$ we get Δx and form the table, that is:

$$\Delta x = \frac{x_{n+1} - x_0}{n+1} = \frac{x_{4+1} - x_0}{4+1} = \frac{x_5 - x_0}{5} = \frac{1-0}{5} = \frac{1}{5}$$

x_0	x_1	x_2	x_3	x_4	x_5
0	0.2	0.4	0.6	0.8	1
1	?	?	?	?	2
y_0	y_1	y_2	y_3	y_4	y_5

Now, we use Eq. 148 with $n = 4$ and $F = y'^2 - y^2 - 2xy$, that is:

$$\begin{aligned}
S\left[y_1, y_2, y_3, y_4\right] &= \sum_{i=0}^{4} \left[\left(y_i'\right)^2 - y_i^2 - 2x_i y_i\right] \Delta x \\
&= \sum_{i=0}^{4} \left[\left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2 - y_i^2 - 2x_i y_i\right] \Delta x \\
&= \frac{1}{5}\Bigg[25\left(y_1 - y_0\right)^2 - y_0^2 - 2x_0 y_0 + 25\left(y_2 - y_1\right)^2 - y_1^2 - 2x_1 y_1 + 25\left(y_3 - y_2\right)^2 \\
&\qquad -y_2^2 - 2x_2 y_2 + 25\left(y_4 - y_3\right)^2 - y_3^2 - 2x_3 y_3 + 25\left(y_5 - y_4\right)^2 - y_4^2 - 2x_4 y_4\Bigg] \\
&= \frac{1}{5}\Bigg[25\left(y_1 - 1\right)^2 - 1 - 0 + 25\left(y_2 - y_1\right)^2 - y_1^2 - \frac{2}{5}y_1 + 25\left(y_3 - y_2\right)^2 \\
&\qquad -y_2^2 - \frac{4}{5}y_2 + 25\left(y_4 - y_3\right)^2 - y_3^2 - \frac{6}{5}y_3 + 25\left(2 - y_4\right)^2 - y_4^2 - \frac{8}{5}y_4\Bigg] \\
&= \frac{49}{5}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2\right) - 10\left(y_1 y_2 + y_2 y_3 + y_3 y_4\right) \\
&\qquad -\frac{1}{25}\left(252 y_1 + 4 y_2 + 6 y_3 + 508 y_4\right) + \frac{124}{5}
\end{aligned}$$

On taking the partial derivatives of S with respect to y_1, y_2, y_3, y_4 and setting the results to zero we get:

$$\begin{aligned}
\frac{\partial S}{\partial y_1} &= \frac{98}{5}y_1 - 10y_2 - \frac{252}{25} = 0 \\
\frac{\partial S}{\partial y_2} &= \frac{98}{5}y_2 - 10y_1 - 10y_3 - \frac{4}{25} = 0 \\
\frac{\partial S}{\partial y_3} &= \frac{98}{5}y_3 - 10y_2 - 10y_4 - \frac{6}{25} = 0 \\
\frac{\partial S}{\partial y_4} &= \frac{98}{5}y_4 - 10y_3 - \frac{508}{25} = 0
\end{aligned}$$

On solving this system of simultaneous equations we get:

$$y_1 = \frac{11255699}{8267755} \qquad y_2 = \frac{13727273}{8267755} \qquad y_3 = \frac{15517472}{8267755} \qquad y_4 = \frac{16488546}{8267755}$$

On plotting the 6 points (i.e. 2 boundaries and 4 inserted) beside the analytical solution (which we obtained in Problem 12 of § 1.4) we get Figure 79.

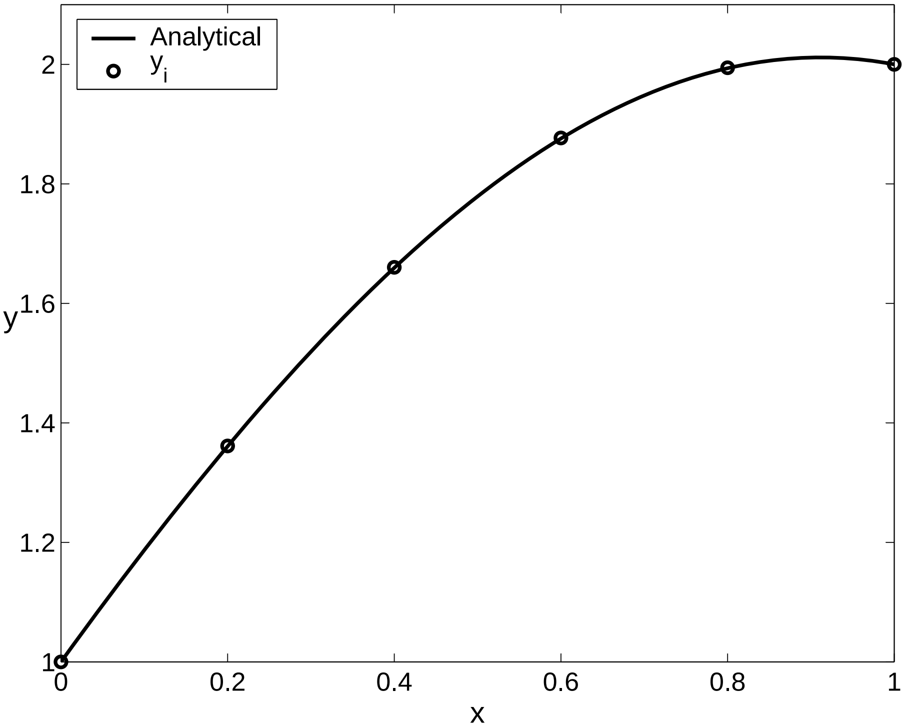

Figure 79: Plot of the analytical solution $y = \cos x + \left(\frac{3-\cos 1}{\sin 1}\right)\sin x - x$ of Problem 4 of § 9 (as obtained in part g of Problem 12 of § 1.4) alongside the points y_1, y_2, y_3, y_4 obtained by finite difference (as well as the boundary points).

5. Re-solve part (f) of Problem 12 of § 1.4 using this time the finite difference method with $n = 5$ discretization scheme (i.e. by inserting 5 evenly-spaced points between the boundaries and hence making 6 divisions). Plot the obtained approximation points beside the analytical solution that you obtained in Problem 12 of § 1.4.
   Answer: From the required discretization scheme $n = 5$ and the boundary conditions $y_0\left(x_0 = 0\right) = 1$ and $y_6\left(x_6 = \pi\right) = 1$ we get Δx and form the table, that is:

$$\Delta x = \frac{x_{n+1} - x_0}{n+1} = \frac{x_{5+1} - x_0}{5+1} = \frac{x_6 - x_0}{6} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

x_0	x_1	x_2	x_3	x_4	x_5	x_6
0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
1	?	?	?	?	?	1
y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, we use Eq. 148 with $n = 5$ and $F = y'^2 + y^2 - 4y\cos x$, that is:

$$\begin{aligned} S\left[y_1, \cdots, y_5\right] &= \sum_{i=0}^{5}\left[\left(y_i'\right)^2 + y_i^2 - 4y_i\cos x_i\right]\Delta x \\ &= \sum_{i=0}^{5}\left[\left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2 + y_i^2 - 4y_i\cos x_i\right]\Delta x \\ &= \sum_{i=0}^{5}\left[\frac{y_{i+1}^2 - 2y_{i+1}y_i + y_i^2}{\Delta x} + y_i^2\Delta x - 4y_i\cos x_i\Delta x\right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{5} \left[\frac{y_{i+1}^2 - 2y_{i+1}y_i + y_i^2 + y_i^2 \left(\Delta x\right)^2 - 4y_i \cos x_i \left(\Delta x\right)^2}{\Delta x} \right] \\
&= \frac{1}{\Delta x} \sum_{i=0}^{5} \left[y_{i+1}^2 - 2y_{i+1}y_i + y_i^2 + y_i^2 \left(\Delta x\right)^2 - 4y_i \cos x_i \left(\Delta x\right)^2 \right] \\
&= \frac{6}{\pi} \sum_{i=0}^{5} \left[y_{i+1}^2 - 2y_{i+1}y_i + y_i^2 + \frac{\pi^2}{36} y_i^2 - \frac{\pi^2}{9} y_i \cos x_i \right] \\
&= \frac{6}{\pi} \Big[y_1^2 - 2y_1y_0 + y_0^2 + \frac{\pi^2}{36} y_0^2 - \frac{\pi^2}{9} y_0 \cos x_0 + y_2^2 - 2y_2y_1 + y_1^2 + \frac{\pi^2}{36} y_1^2 - \frac{\pi^2}{9} y_1 \cos x_1 \\
&\quad + y_3^2 - 2y_3y_2 + y_2^2 + \frac{\pi^2}{36} y_2^2 - \frac{\pi^2}{9} y_2 \cos x_2 + y_4^2 - 2y_4y_3 + y_3^2 + \frac{\pi^2}{36} y_3^2 - \frac{\pi^2}{9} y_3 \cos x_3 \\
&\quad + y_5^2 - 2y_5y_4 + y_4^2 + \frac{\pi^2}{36} y_4^2 - \frac{\pi^2}{9} y_4 \cos x_4 + y_6^2 - 2y_6y_5 + y_5^2 + \frac{\pi^2}{36} y_5^2 - \frac{\pi^2}{9} y_5 \cos x_5 \Big] \\
&= \frac{6}{\pi} \Big[y_1^2 - 2y_1 + 1 + \frac{\pi^2}{36} - \frac{\pi^2}{9} + y_2^2 - 2y_2y_1 + y_1^2 + \frac{\pi^2}{36} y_1^2 - \frac{\pi^2\sqrt{3}}{18} y_1 \\
&\quad + y_3^2 - 2y_3y_2 + y_2^2 + \frac{\pi^2}{36} y_2^2 - \frac{\pi^2}{18} y_2 + y_4^2 - 2y_4y_3 + y_3^2 + \frac{\pi^2}{36} y_3^2 \\
&\quad + y_5^2 - 2y_5y_4 + y_4^2 + \frac{\pi^2}{36} y_4^2 + \frac{\pi^2}{18} y_4 + 1 - 2y_5 + y_5^2 + \frac{\pi^2}{36} y_5^2 + \frac{\pi^2\sqrt{3}}{18} y_5 \Big] \\
&= \frac{6}{\pi} \left[\left(2 + \frac{\pi^2}{36}\right) \left(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\right) - 2\left(y_1y_2 + y_2y_3 + y_3y_4 + y_4y_5\right) \right. \\
&\quad \left. - \left(2 + \frac{\pi^2\sqrt{3}}{18}\right) y_1 - \frac{\pi^2}{18} y_2 + \frac{\pi^2}{18} y_4 - \left(2 - \frac{\pi^2\sqrt{3}}{18}\right) y_5 + 2 - \frac{\pi^2}{12} \right]
\end{aligned}$$

On taking the partial derivatives of S with respect to y_1, y_2, y_3, y_4, y_5 and setting the results to zero we get:

$$\begin{aligned}
\frac{\partial S}{\partial y_1} &= \left(4 + \frac{\pi^2}{18}\right) y_1 - 2y_2 - \left(2 + \frac{\pi^2\sqrt{3}}{18}\right) = 0 \\
\frac{\partial S}{\partial y_2} &= \left(4 + \frac{\pi^2}{18}\right) y_2 - 2y_1 - 2y_3 - \frac{\pi^2}{18} = 0 \\
\frac{\partial S}{\partial y_3} &= \left(4 + \frac{\pi^2}{18}\right) y_3 - 2y_2 - 2y_4 = 0 \\
\frac{\partial S}{\partial y_4} &= \left(4 + \frac{\pi^2}{18}\right) y_4 - 2y_3 - 2y_5 + \frac{\pi^2}{18} = 0 \\
\frac{\partial S}{\partial y_5} &= \left(4 + \frac{\pi^2}{18}\right) y_5 - 2y_4 - \left(2 - \frac{\pi^2\sqrt{3}}{18}\right) = 0
\end{aligned}$$

On solving this system of simultaneous equations we get:

$$y_1 \simeq 0.96676626 \qquad y_2 \simeq 0.72372541 \qquad y_3 \simeq 0.40494232 \qquad y_4 \simeq 0.19717646 \qquad y_5 \simeq 0.31762333$$

On plotting the 7 points (i.e. 2 boundaries and 5 inserted) beside the analytical solution (which we obtained in Problem 12 of § 1.4) we get Figure 80.

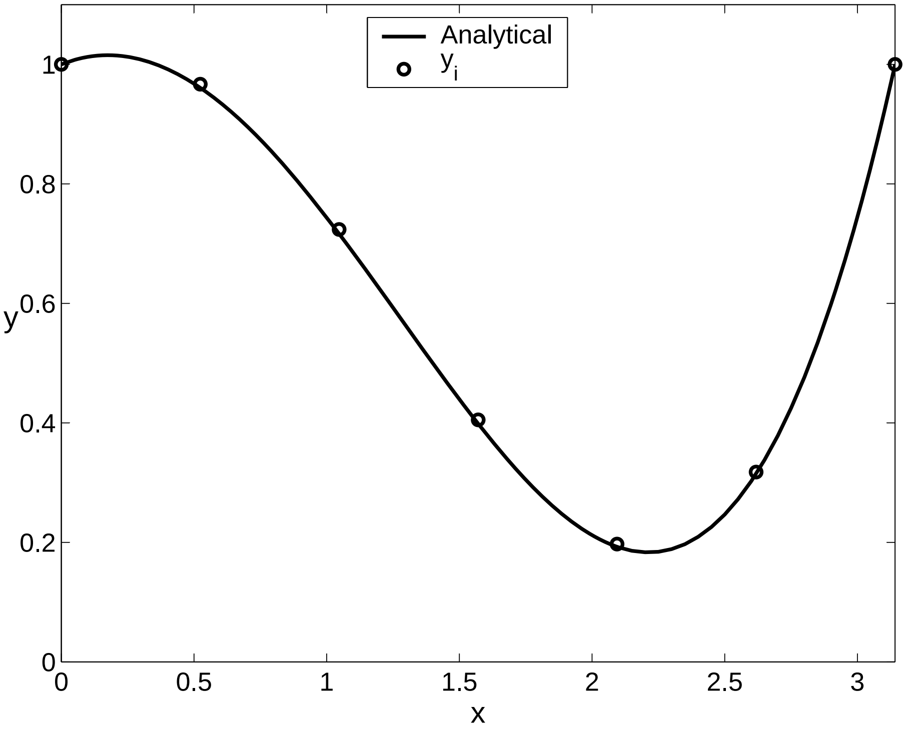

Figure 80: Plot of the analytical solution $y = \left(\frac{2}{\sinh \pi}\right) \sinh x + \cos x$ of Problem 5 of § 9 (as obtained in part f of Problem 12 of § 1.4) alongside the points y_1, y_2, y_3, y_4, y_5 obtained by finite difference (as well as the boundary points).

6. Re-solve Problem 7 of § 1.6 using this time the finite difference method with $m = n = 3$ discretization scheme (i.e. by inserting 3 evenly-spaced points between the x and y boundaries and hence making 16 square divisions). Plot the obtained approximation points beside the analytical solution that you obtained in Problem 7 of § 1.6.
Answer: This is a 2D problem and hence in the following we will extend (rather briefly) the above-described 1D finite difference method to 2D. From the required discretization scheme $m = n = 3$ and the boundary conditions $z(0, y) = z(x, 0) = z(1, y) = 0$ and $z(x, 1) = \sin(\pi x) \sinh(\pi)$ we get Δx and Δy and form the table, that is:

$$\Delta x = \frac{x_{m+1} - x_0}{m+1} = \frac{1-0}{3+1} = \frac{1}{4} \qquad \text{and} \qquad \Delta y = \frac{y_{n+1} - y_0}{n+1} = \frac{1-0}{3+1} = \frac{1}{4}$$

	$x_0 = 0$	$x_1 = \frac{1}{4}$	$x_2 = \frac{2}{4}$	$x_3 = \frac{3}{4}$	$x_4 = \frac{4}{4}$
$y_0 = 0$	$z_{00} = 0$	$z_{10} = 0$	$z_{20} = 0$	$z_{30} = 0$	$z_{40} = 0$
$y_1 = \frac{1}{4}$	$z_{01} = 0$	$z_{11} = ?$	$z_{21} = ?$	$z_{31} = ?$	$z_{41} = 0$
$y_2 = \frac{2}{4}$	$z_{02} = 0$	$z_{12} = ?$	$z_{22} = ?$	$z_{32} = ?$	$z_{42} = 0$
$y_3 = \frac{3}{4}$	$z_{03} = 0$	$z_{13} = ?$	$z_{23} = ?$	$z_{33} = ?$	$z_{43} = 0$
$y_4 = \frac{4}{4}$	$z_{04} = 0$	$z_{14} = \sin\left(\frac{\pi}{4}\right) \sinh(\pi)$	$z_{24} = \sinh(\pi)$	$z_{34} = \sin\left(\frac{3\pi}{4}\right) \sinh(\pi)$	$z_{44} = 0$

Now, we should generalize Eq. 148 to 2D, that is:

$$S[z_{11}, \cdots, z_{mn}] = \sum_{j=0}^{n} \sum_{i=0}^{m} F(x_i, y_j, z_{ij}, z_{ij,x}, z_{ij,y}) \, \Delta x \, \Delta y \tag{150}$$

where

$$z_{ij,x} = \frac{z_{(i+1)j} - z_{ij}}{\Delta x} \qquad \text{and} \qquad z_{ij,y} = \frac{z_{i(j+1)} - z_{ij}}{\Delta y} \tag{151}$$

On using this equation with $m = n = 3$ and $F = z_x^2 + z_y^2$, we get:

$$\begin{aligned}
S\left[z_{11}, \cdots, z_{33}\right] &= \sum_{j=0}^{3}\sum_{i=0}^{3}\left[z_{ij,x}^2 + z_{ij,y}^2\right]\Delta x\,\Delta y \\
&= \sum_{j=0}^{3}\sum_{i=0}^{3}\left[\left(\frac{z_{(i+1)j} - z_{ij}}{\Delta x}\right)^2 + \left(\frac{z_{i(j+1)} - z_{ij}}{\Delta y}\right)^2\right]\Delta x\,\Delta y \\
&= \sum_{j=0}^{3}\sum_{i=0}^{3}\left[\frac{z_{(i+1)j}^2 - 2z_{(i+1)j}z_{ij} + z_{ij}^2}{\Delta x \Delta x} + \frac{z_{i(j+1)}^2 - 2z_{i(j+1)}z_{ij} + z_{ij}^2}{\Delta y \Delta y}\right]\Delta x\,\Delta y
\end{aligned}$$

Now, if we have $\Delta x = \Delta y$ (as in our case) then this formula will simplify to the following:

$$S = \sum_{j=0}^{3}\sum_{i=0}^{3}\left[z_{(i+1)j}^2 - 2z_{(i+1)j}z_{ij} + z_{i(j+1)}^2 - 2z_{i(j+1)}z_{ij} + 2z_{ij}^2\right]$$

that is:

$$\begin{array}{rcll}
S &=& z_{10}^2 - 2z_{10}z_{00} + z_{01}^2 - 2z_{01}z_{00} + 2z_{00}^2 + & (i=0, j=0) \\
&& z_{20}^2 - 2z_{20}z_{10} + z_{11}^2 - 2z_{11}z_{10} + 2z_{10}^2 + & (i=1, j=0) \\
&& z_{30}^2 - 2z_{30}z_{20} + z_{21}^2 - 2z_{21}z_{20} + 2z_{20}^2 + & (i=2, j=0) \\
&& z_{40}^2 - 2z_{40}z_{30} + z_{31}^2 - 2z_{31}z_{30} + 2z_{30}^2 + & (i=3, j=0) \\
&& z_{11}^2 - 2z_{11}z_{01} + z_{02}^2 - 2z_{02}z_{01} + 2z_{01}^2 + & (i=0, j=1) \\
&& z_{21}^2 - 2z_{21}z_{11} + z_{12}^2 - 2z_{12}z_{11} + 2z_{11}^2 + & (i=1, j=1) \\
&& z_{31}^2 - 2z_{31}z_{21} + z_{22}^2 - 2z_{22}z_{21} + 2z_{21}^2 + & (i=2, j=1) \\
&& z_{41}^2 - 2z_{41}z_{31} + z_{32}^2 - 2z_{32}z_{31} + 2z_{31}^2 + & (i=3, j=1) \\
&& z_{12}^2 - 2z_{12}z_{02} + z_{03}^2 - 2z_{03}z_{02} + 2z_{02}^2 + & (i=0, j=2) \\
&& z_{22}^2 - 2z_{22}z_{12} + z_{13}^2 - 2z_{13}z_{12} + 2z_{12}^2 + & (i=1, j=2) \\
&& z_{32}^2 - 2z_{32}z_{22} + z_{23}^2 - 2z_{23}z_{22} + 2z_{22}^2 + & (i=2, j=2) \\
&& z_{42}^2 - 2z_{42}z_{32} + z_{33}^2 - 2z_{33}z_{32} + 2z_{32}^2 + & (i=3, j=2) \\
&& z_{13}^2 - 2z_{13}z_{03} + z_{04}^2 - 2z_{04}z_{03} + 2z_{03}^2 + & (i=0, j=3) \\
&& z_{23}^2 - 2z_{23}z_{13} + z_{14}^2 - 2z_{14}z_{13} + 2z_{13}^2 + & (i=1, j=3) \\
&& z_{33}^2 - 2z_{33}z_{23} + z_{24}^2 - 2z_{24}z_{23} + 2z_{23}^2 + & (i=2, j=3) \\
&& z_{43}^2 - 2z_{43}z_{33} + z_{34}^2 - 2z_{34}z_{33} + 2z_{33}^2 & (i=3, j=3)
\end{array}$$

On combining similar terms and applying the first three boundary conditions $\left[\text{i.e. } z(0,y) = z(x,0) = z(1,y) = 0\right]$ by eliminating all the terms containing z_{0j} or z_{i0} or z_{4j}, this expression of S will simplify to the following:

$$\begin{aligned}
S \;=\;& 4z_{11}^2 + 4z_{21}^2 + 4z_{31}^2 + 4z_{12}^2 + 4z_{22}^2 + 4z_{32}^2 + 4z_{13}^2 + 4z_{23}^2 + 4z_{33}^2 + \\
& z_{14}^2 + z_{24}^2 + z_{34}^2 - 2z_{21}z_{11} - 2z_{12}z_{11} - 2z_{31}z_{21} - 2z_{22}z_{21} - 2z_{32}z_{31} - 2z_{22}z_{12} - 2z_{13}z_{12} \\
& -2z_{32}z_{22} - 2z_{23}z_{22} - 2z_{33}z_{32} - 2z_{23}z_{13} - 2z_{14}z_{13} - 2z_{33}z_{23} - 2z_{24}z_{23} - 2z_{34}z_{33}
\end{aligned}$$

On taking the partial derivatives of S with respect to $z_{11}, \cdots, z_{33}$ and setting the results to zero (with the application of the fourth boundary condition which concerns z_{14}, z_{24}, z_{34}) we get:

$$\frac{\partial S}{\partial z_{11}} = 8z_{11} - 2z_{21} - 2z_{12} = 0$$

$$\begin{aligned}
\frac{\partial S}{\partial z_{12}} &= 8z_{12} - 2z_{11} - 2z_{22} - 2z_{13} = 0 \\
\frac{\partial S}{\partial z_{13}} &= 8z_{13} - 2z_{12} - 2z_{23} - 2\sin\left(\frac{\pi}{4}\right)\sinh(\pi) = 0 \\
\frac{\partial S}{\partial z_{21}} &= 8z_{21} - 2z_{11} - 2z_{31} - 2z_{22} = 0 \\
\frac{\partial S}{\partial z_{22}} &= 8z_{22} - 2z_{21} - 2z_{12} - 2z_{32} - 2z_{23} = 0 \\
\frac{\partial S}{\partial z_{23}} &= 8z_{23} - 2z_{22} - 2z_{13} - 2z_{33} - 2\sinh(\pi) = 0 \\
\frac{\partial S}{\partial z_{31}} &= 8z_{31} - 2z_{21} - 2z_{32} = 0 \\
\frac{\partial S}{\partial z_{32}} &= 8z_{32} - 2z_{31} - 2z_{22} - 2z_{33} = 0 \\
\frac{\partial S}{\partial z_{33}} &= 8z_{33} - 2z_{32} - 2z_{23} - 2\sin\left(\frac{3\pi}{4}\right)\sinh(\pi) = 0
\end{aligned}$$

On solving this system of simultaneous equations we get:

$z_{11} \simeq 0.673903372$	$z_{21} \simeq 0.953043288$	$z_{31} \simeq 0.673903372$
$z_{12} \simeq 1.742570199$	$z_{22} \simeq 2.464366408$	$z_{32} \simeq 1.742570199$
$z_{13} \simeq 3.832011015$	$z_{23} \simeq 5.419281947$	$z_{33} \simeq 3.832011015$

On plotting these points with the boundary points (see the upper frame of Figure 81) alongside the analytical solution (see the lower frame of Figure 81) which we obtained in Problem 7 of § 1.6 we see that the two plots are almost identical.

7. Re-solve Problem 8 of § 1.6 using this time the finite difference method with $m = n = 4$ discretization scheme (i.e. by inserting 4 evenly-spaced points between the boundaries and hence making 25 square divisions). Plot the obtained approximation points beside the analytical solution that you obtained in Problem 8 of § 1.6.

Answer: From the required discretization scheme $m = n = 4$ and the boundary conditions $z(0, y) = z(x, 0) = z(1, y) = 0$ and $z(x, 1) = \sin(\pi x)$ we get Δx and Δy and form the table, that is:

$$\Delta x = \frac{x_{m+1} - x_0}{m+1} = \frac{1-0}{4+1} = \frac{1}{5} \qquad \text{and} \qquad \Delta y = \frac{y_{n+1} - y_0}{n+1} = \frac{1-0}{4+1} = \frac{1}{5}$$

	$x_0 = 0$	$x_1 = \frac{1}{5}$	$x_2 = \frac{2}{5}$	$x_3 = \frac{3}{5}$	$x_4 = \frac{4}{5}$	$x_5 = \frac{5}{5}$
$y_0 = 0$	$z_{00} = 0$	$z_{10} = 0$	$z_{20} = 0$	$z_{30} = 0$	$z_{40} = 0$	$z_{50} = 0$
$y_1 = \frac{1}{5}$	$z_{01} = 0$	$z_{11} = ?$	$z_{21} = ?$	$z_{31} = ?$	$z_{41} = ?$	$z_{51} = 0$
$y_2 = \frac{2}{5}$	$z_{02} = 0$	$z_{12} = ?$	$z_{22} = ?$	$z_{32} = ?$	$z_{42} = ?$	$z_{52} = 0$
$y_3 = \frac{3}{5}$	$z_{03} = 0$	$z_{13} = ?$	$z_{23} = ?$	$z_{33} = ?$	$z_{43} = ?$	$z_{53} = 0$
$y_4 = \frac{4}{5}$	$z_{04} = 0$	$z_{14} = ?$	$z_{24} = ?$	$z_{34} = ?$	$z_{44} = ?$	$z_{54} = 0$
$y_5 = \frac{5}{5}$	$z_{05} = 0$	$z_{15} = \sin\left(\frac{\pi}{5}\right)$	$z_{25} = \sin\left(\frac{2\pi}{5}\right)$	$z_{35} = \sin\left(\frac{3\pi}{5}\right)$	$z_{45} = \sin\left(\frac{4\pi}{5}\right)$	$z_{55} = 0$

On generalizing Eq. 148 to 2D (as we did in Problem 6; see Eqs. 150 and 151) and noting that $m = n = 4$ and $F = z_x^2 + z_y^2 + \left(2y^{-2} - \pi^2\right) z^2$ we get $S[z_{11}, \cdots, z_{44}]$, that is:

$$\begin{aligned}
S &= \sum_{j=0}^{4} \sum_{i=0}^{4} \left[z_{ij,x}^2 + z_{ij,y}^2 + \left(2y_j^{-2} - \pi^2\right) z_{ij}^2 \right] \Delta x \, \Delta y \\
&= \sum_{j=0}^{4} \sum_{i=0}^{4} \left[\left(\frac{z_{(i+1)j} - z_{ij}}{\Delta x} \right)^2 + \left(\frac{z_{i(j+1)} - z_{ij}}{\Delta y} \right)^2 + \left(2y_j^{-2} - \pi^2\right) z_{ij}^2 \right] \Delta x \, \Delta y
\end{aligned}$$

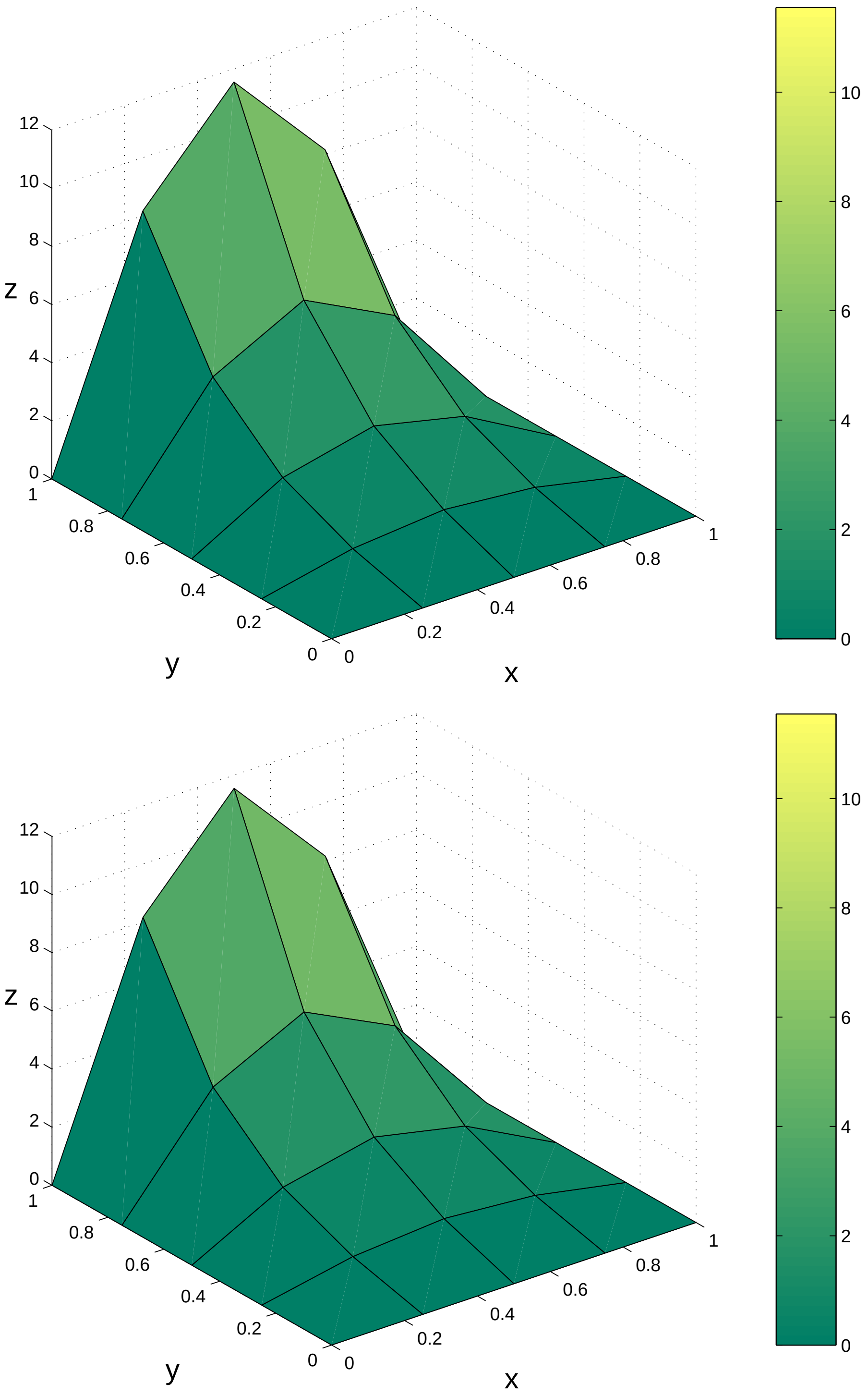

Figure 81: The upper frame is a plot of the points $z_{11}, \cdots, z_{33}$ which we obtained in Problem 6 of § 9 by finite difference (as well as the boundary points) while the lower frame is a plot of the analytical solution $z = \sin(\pi x) \sinh(\pi y)$ which we obtained in Problem 7 of § 1.6. For fair comparison, we use the same xy mesh in both plots.

$$= \sum_{j=0}^{4}\sum_{i=0}^{4}\left[\frac{z^2_{(i+1)j} - 2z_{(i+1)j}z_{ij} + z^2_{ij}}{\Delta x\Delta x} + \frac{z^2_{i(j+1)} - 2z_{i(j+1)}z_{ij} + z^2_{ij}}{\Delta y\Delta y} + \left(2y_j^{-2} - \pi^2\right) z^2_{ij}\right]\Delta x\,\Delta y$$

Now, if we have $\Delta x = \Delta y = 0.2$ (as in our case) then this formula will simplify to the following:

$$\begin{aligned}
S &= \sum_{j=0}^{4}\sum_{i=0}^{4}\left[z^2_{(i+1)j} - 2z_{(i+1)j}z_{ij} + z^2_{i(j+1)} - 2z_{i(j+1)}z_{ij} + 2z^2_{ij} + 0.04\left(2y_j^{-2} - \pi^2\right) z^2_{ij}\right]\\
&= \sum_{j=0}^{4}\sum_{i=0}^{4}\left[z^2_{(i+1)j} - 2z_{(i+1)j}z_{ij} + z^2_{i(j+1)} - 2z_{i(j+1)}z_{ij} + \left(0.08y_j^{-2} + 2 - 0.04\pi^2\right) z^2_{ij}\right]
\end{aligned}$$

that is:

$$\begin{aligned}
S = {} & z^2_{10} - 2z_{10}z_{00} + z^2_{01} - 2z_{01}z_{00} + \left(0.08y_0^{-2} + 2 - 0.04\pi^2\right) z^2_{00} + && (i = 0, j = 0)\\
& z^2_{20} - 2z_{20}z_{10} + z^2_{11} - 2z_{11}z_{10} + \left(0.08y_0^{-2} + 2 - 0.04\pi^2\right) z^2_{10} + && (i = 1, j = 0)\\
& z^2_{30} - 2z_{30}z_{20} + z^2_{21} - 2z_{21}z_{20} + \left(0.08y_0^{-2} + 2 - 0.04\pi^2\right) z^2_{20} + && (i = 2, j = 0)\\
& z^2_{40} - 2z_{40}z_{30} + z^2_{31} - 2z_{31}z_{30} + \left(0.08y_0^{-2} + 2 - 0.04\pi^2\right) z^2_{30} + && (i = 3, j = 0)\\
& z^2_{50} - 2z_{50}z_{40} + z^2_{41} - 2z_{41}z_{40} + \left(0.08y_0^{-2} + 2 - 0.04\pi^2\right) z^2_{40} + && (i = 4, j = 0)\\
& z^2_{11} - 2z_{11}z_{01} + z^2_{02} - 2z_{02}z_{01} + \left(0.08y_1^{-2} + 2 - 0.04\pi^2\right) z^2_{01} + && (i = 0, j = 1)\\
& z^2_{21} - 2z_{21}z_{11} + z^2_{12} - 2z_{12}z_{11} + \left(0.08y_1^{-2} + 2 - 0.04\pi^2\right) z^2_{11} + && (i = 1, j = 1)\\
& z^2_{31} - 2z_{31}z_{21} + z^2_{22} - 2z_{22}z_{21} + \left(0.08y_1^{-2} + 2 - 0.04\pi^2\right) z^2_{21} + && (i = 2, j = 1)\\
& z^2_{41} - 2z_{41}z_{31} + z^2_{32} - 2z_{32}z_{31} + \left(0.08y_1^{-2} + 2 - 0.04\pi^2\right) z^2_{31} + && (i = 3, j = 1)\\
& z^2_{51} - 2z_{51}z_{41} + z^2_{42} - 2z_{42}z_{41} + \left(0.08y_1^{-2} + 2 - 0.04\pi^2\right) z^2_{41} + && (i = 4, j = 1)\\
& z^2_{12} - 2z_{12}z_{02} + z^2_{03} - 2z_{03}z_{02} + \left(0.08y_2^{-2} + 2 - 0.04\pi^2\right) z^2_{02} + && (i = 0, j = 2)\\
& z^2_{22} - 2z_{22}z_{12} + z^2_{13} - 2z_{13}z_{12} + \left(0.08y_2^{-2} + 2 - 0.04\pi^2\right) z^2_{12} + && (i = 1, j = 2)\\
& z^2_{32} - 2z_{32}z_{22} + z^2_{23} - 2z_{23}z_{22} + \left(0.08y_2^{-2} + 2 - 0.04\pi^2\right) z^2_{22} + && (i = 2, j = 2)\\
& z^2_{42} - 2z_{42}z_{32} + z^2_{33} - 2z_{33}z_{32} + \left(0.08y_2^{-2} + 2 - 0.04\pi^2\right) z^2_{32} + && (i = 3, j = 2)\\
& z^2_{52} - 2z_{52}z_{42} + z^2_{43} - 2z_{43}z_{42} + \left(0.08y_2^{-2} + 2 - 0.04\pi^2\right) z^2_{42} + && (i = 4, j = 2)\\
& z^2_{13} - 2z_{13}z_{03} + z^2_{04} - 2z_{04}z_{03} + \left(0.08y_3^{-2} + 2 - 0.04\pi^2\right) z^2_{03} + && (i = 0, j = 3)\\
& z^2_{23} - 2z_{23}z_{13} + z^2_{14} - 2z_{14}z_{13} + \left(0.08y_3^{-2} + 2 - 0.04\pi^2\right) z^2_{13} + && (i = 1, j = 3)\\
& z^2_{33} - 2z_{33}z_{23} + z^2_{24} - 2z_{24}z_{23} + \left(0.08y_3^{-2} + 2 - 0.04\pi^2\right) z^2_{23} + && (i = 2, j = 3)\\
& z^2_{43} - 2z_{43}z_{33} + z^2_{34} - 2z_{34}z_{33} + \left(0.08y_3^{-2} + 2 - 0.04\pi^2\right) z^2_{33} + && (i = 3, j = 3)\\
& z^2_{53} - 2z_{53}z_{43} + z^2_{44} - 2z_{44}z_{43} + \left(0.08y_3^{-2} + 2 - 0.04\pi^2\right) z^2_{43} + && (i = 4, j = 3)\\
& z^2_{14} - 2z_{14}z_{04} + z^2_{05} - 2z_{05}z_{04} + \left(0.08y_4^{-2} + 2 - 0.04\pi^2\right) z^2_{04} + && (i = 0, j = 4)\\
& z^2_{24} - 2z_{24}z_{14} + z^2_{15} - 2z_{15}z_{14} + \left(0.08y_4^{-2} + 2 - 0.04\pi^2\right) z^2_{14} + && (i = 1, j = 4)\\
& z^2_{34} - 2z_{34}z_{24} + z^2_{25} - 2z_{25}z_{24} + \left(0.08y_4^{-2} + 2 - 0.04\pi^2\right) z^2_{24} + && (i = 2, j = 4)\\
& z^2_{44} - 2z_{44}z_{34} + z^2_{35} - 2z_{35}z_{34} + \left(0.08y_4^{-2} + 2 - 0.04\pi^2\right) z^2_{34} + && (i = 3, j = 4)\\
& z^2_{54} - 2z_{54}z_{44} + z^2_{45} - 2z_{45}z_{44} + \left(0.08y_4^{-2} + 2 - 0.04\pi^2\right) z^2_{44} && (i = 4, j = 4)
\end{aligned}$$

On combining similar terms and applying the first three boundary conditions $\left[\text{i.e. } z\left(0, y\right) = z\left(x, 0\right) = z\left(1, y\right) = 0\right]$ by eliminating all the terms containing z_{0j} or z_{i0} or z_{5j} as well as inserting the numeric values, this expression of S will simplify to the following:

$$S = -2z_{21}z_{11} - 2z_{12}z_{11} + \left(6 - 0.04\pi^2\right) z^2_{11}$$

$$
\begin{aligned}
&-2z_{31}z_{21} - 2z_{22}z_{21} + \left(6 - 0.04\pi^2\right) z_{21}^2 \\
&-2z_{41}z_{31} - 2z_{32}z_{31} + \left(6 - 0.04\pi^2\right) z_{31}^2 \\
&-2z_{42}z_{41} + \left(6 - 0.04\pi^2\right) z_{41}^2 \\
&-2z_{22}z_{12} - 2z_{13}z_{12} + \left(4.5 - 0.04\pi^2\right) z_{12}^2 \\
&-2z_{32}z_{22} - 2z_{23}z_{22} + \left(4.5 - 0.04\pi^2\right) z_{22}^2 \\
&-2z_{42}z_{32} - 2z_{33}z_{32} + \left(4.5 - 0.04\pi^2\right) z_{32}^2 \\
&-2z_{43}z_{42} + \left(4.5 - 0.04\pi^2\right) z_{42}^2 \\
&-2z_{23}z_{13} - 2z_{14}z_{13} + \left(4.222222 - 0.04\pi^2\right) z_{13}^2 \\
&-2z_{33}z_{23} - 2z_{24}z_{23} + \left(4.222222 - 0.04\pi^2\right) z_{23}^2 \\
&-2z_{43}z_{33} - 2z_{34}z_{33} + \left(4.222222 - 0.04\pi^2\right) z_{33}^2 \\
&-2z_{44}z_{43} + \left(4.222222 - 0.04\pi^2\right) z_{43}^2 \\
&-2z_{24}z_{14} + \sin^2\left(\frac{\pi}{5}\right) - 2\sin\left(\frac{\pi}{5}\right) z_{14} + \left(4.125 - 0.04\pi^2\right) z_{14}^2 \\
&-2z_{34}z_{24} + \sin^2\left(\frac{2\pi}{5}\right) - 2\sin\left(\frac{2\pi}{5}\right) z_{24} + \left(4.125 - 0.04\pi^2\right) z_{24}^2 \\
&-2z_{44}z_{34} + \sin^2\left(\frac{3\pi}{5}\right) - 2\sin\left(\frac{3\pi}{5}\right) z_{34} + \left(4.125 - 0.04\pi^2\right) z_{34}^2 \\
&+\sin^2\left(\frac{4\pi}{5}\right) - 2\sin\left(\frac{4\pi}{5}\right) z_{44} + \left(4.125 - 0.04\pi^2\right) z_{44}^2
\end{aligned}
$$

On taking the partial derivatives of S with respect to $z_{11}, \cdots, z_{44}$ and setting the results to zero we get:

$$
\begin{aligned}
\frac{\partial S}{\partial z_{11}} &= -2z_{21} - 2z_{12} + 2\left(6 - 0.04\pi^2\right) z_{11} = 0 \\
\frac{\partial S}{\partial z_{12}} &= -2z_{11} - 2z_{22} - 2z_{13} + 2\left(4.5 - 0.04\pi^2\right) z_{12} = 0 \\
\frac{\partial S}{\partial z_{13}} &= -2z_{12} - 2z_{23} - 2z_{14} + 2\left(4.222222 - 0.04\pi^2\right) z_{13} = 0 \\
\frac{\partial S}{\partial z_{14}} &= -2z_{13} - 2z_{24} - 2\sin\left(\frac{\pi}{5}\right) + 2\left(4.125 - 0.04\pi^2\right) z_{14} = 0 \\
\frac{\partial S}{\partial z_{21}} &= -2z_{11} - 2z_{31} - 2z_{22} + 2\left(6 - 0.04\pi^2\right) z_{21} = 0 \\
\frac{\partial S}{\partial z_{22}} &= -2z_{21} - 2z_{12} - 2z_{32} - 2z_{23} + 2\left(4.5 - 0.04\pi^2\right) z_{22} = 0 \\
\frac{\partial S}{\partial z_{23}} &= -2z_{22} - 2z_{13} - 2z_{33} - 2z_{24} + 2\left(4.222222 - 0.04\pi^2\right) z_{23} = 0 \\
\frac{\partial S}{\partial z_{24}} &= -2z_{23} - 2z_{14} - 2z_{34} - 2\sin\left(\frac{2\pi}{5}\right) + 2\left(4.125 - 0.04\pi^2\right) z_{24} = 0 \\
\frac{\partial S}{\partial z_{31}} &= -2z_{21} - 2z_{41} - 2z_{32} + 2\left(6 - 0.04\pi^2\right) z_{31} = 0 \\
\frac{\partial S}{\partial z_{32}} &= -2z_{31} - 2z_{22} - 2z_{42} - 2z_{33} + 2\left(4.5 - 0.04\pi^2\right) z_{32} = 0 \\
\frac{\partial S}{\partial z_{33}} &= -2z_{32} - 2z_{23} - 2z_{43} - 2z_{34} + 2\left(4.222222 - 0.04\pi^2\right) z_{33} = 0 \\
\frac{\partial S}{\partial z_{34}} &= -2z_{33} - 2z_{24} - 2z_{44} - 2\sin\left(\frac{3\pi}{5}\right) + 2\left(4.125 - 0.04\pi^2\right) z_{34} = 0
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial S}{\partial z_{41}} &= -2z_{31} - 2z_{42} + 2\left(6 - 0.04\pi^2\right) z_{41} = 0 \\
\frac{\partial S}{\partial z_{42}} &= -2z_{41} - 2z_{32} - 2z_{43} + 2\left(4.5 - 0.04\pi^2\right) z_{42} = 0 \\
\frac{\partial S}{\partial z_{43}} &= -2z_{42} - 2z_{33} - 2z_{44} + 2\left(4.222222 - 0.04\pi^2\right) z_{43} = 0 \\
\frac{\partial S}{\partial z_{44}} &= -2z_{43} - 2z_{34} - 2\sin\left(\frac{4\pi}{5}\right) + 2\left(4.125 - 0.04\pi^2\right) z_{44} = 0
\end{aligned}$$

On solving this system of simultaneous equations we get:

$z_{11} \simeq 0.0242152$	$z_{21} \simeq 0.0391810$	$z_{31} \simeq 0.0391810$	$z_{41} \simeq 0.0242152$
$z_{12} \simeq 0.0965504$	$z_{22} \simeq 0.1562218$	$z_{32} \simeq 0.1562218$	$z_{42} \simeq 0.0965504$
$z_{13} \simeq 0.2159231$	$z_{23} \simeq 0.3493709$	$z_{33} \simeq 0.3493709$	$z_{43} \simeq 0.2159231$
$z_{14} \simeq 0.3805110$	$z_{24} \simeq 0.6156797$	$z_{34} \simeq 0.6156797$	$z_{44} \simeq 0.3805110$

On plotting these points with the boundary points (see the upper frame of Figure 82) alongside the analytical solution (see the lower frame of Figure 82) which we obtained in Problem 8 of § 1.6 we see that the two plots are almost identical.

8. Re-solve Problem 9 of § 1.6 using this time the finite difference method with $m = n = 5$ discretization scheme (i.e. by inserting 5 evenly-spaced points between the boundaries and hence making 36 square divisions).

Answer: From the required discretization scheme $m = n = 5$ and the boundary conditions $z(0, y) = 0$, $z(x, 0) = -x^2$, $z(1, y) = y - 1$ and $z(x, 1) = x - x^2$ we get Δx and Δy and form the table, that is:

$$\Delta x = \frac{x_{m+1} - x_0}{m+1} = \frac{1-0}{5+1} = \frac{1}{6} \qquad \text{and} \qquad \Delta y = \frac{y_{n+1} - y_0}{n+1} = \frac{1-0}{5+1} = \frac{1}{6}$$

	$x_0 = 0$	$x_1 = \frac{1}{6}$	$x_2 = \frac{2}{6}$	$x_3 = \frac{3}{6}$	$x_4 = \frac{4}{6}$	$x_5 = \frac{5}{6}$	$x_6 = \frac{6}{6}$
$y_0 = 0$	$z_{00} = 0$	$z_{10} = -\frac{1}{36}$	$z_{20} = -\frac{4}{36}$	$z_{30} = -\frac{9}{36}$	$z_{40} = -\frac{16}{36}$	$z_{50} = -\frac{25}{36}$	$z_{60} = -1$
$y_1 = \frac{1}{6}$	$z_{01} = 0$	$z_{11} =?$	$z_{21} =?$	$z_{31} =?$	$z_{41} =?$	$z_{51} =?$	$z_{61} = -\frac{5}{6}$
$y_2 = \frac{2}{6}$	$z_{02} = 0$	$z_{12} =?$	$z_{22} =?$	$z_{32} =?$	$z_{42} =?$	$z_{52} =?$	$z_{62} = -\frac{4}{6}$
$y_3 = \frac{3}{6}$	$z_{03} = 0$	$z_{13} =?$	$z_{23} =?$	$z_{33} =?$	$z_{43} =?$	$z_{53} =?$	$z_{63} = -\frac{3}{6}$
$y_4 = \frac{4}{6}$	$z_{04} = 0$	$z_{14} =?$	$z_{24} =?$	$z_{34} =?$	$z_{44} =?$	$z_{54} =?$	$z_{64} = -\frac{2}{6}$
$y_5 = \frac{5}{6}$	$z_{05} = 0$	$z_{15} =?$	$z_{25} =?$	$z_{35} =?$	$z_{45} =?$	$z_{55} =?$	$z_{65} = -\frac{1}{6}$
$y_6 = \frac{6}{6}$	$z_{06} = 0$	$z_{16} = \frac{5}{36}$	$z_{26} = \frac{8}{36}$	$z_{36} = \frac{9}{36}$	$z_{46} = \frac{8}{36}$	$z_{56} = \frac{5}{36}$	$z_{66} = 0$

On generalizing Eq. 148 to 2D (as we did in Problem 6; see Eqs. 150 and 151) and noting that $m = n = 5$ and $F = z_x^2 + z_y^2 - 4z$ we get $S[z_{11}, \cdots, z_{55}]$, that is:

$$\begin{aligned}
S &= \sum_{j=0}^{5}\sum_{i=0}^{5}\left[z_{ij,x}^2 + z_{ij,y}^2 - 4z_{ij}\right]\Delta x\,\Delta y \\
&= \sum_{j=0}^{5}\sum_{i=0}^{5}\left[\left(\frac{z_{(i+1)j} - z_{ij}}{\Delta x}\right)^2 + \left(\frac{z_{i(j+1)} - z_{ij}}{\Delta y}\right)^2 - 4z_{ij}\right]\Delta x\,\Delta y \\
&= \sum_{j=0}^{5}\sum_{i=0}^{5}\left[\frac{z_{(i+1)j}^2 - 2z_{(i+1)j}z_{ij} + z_{ij}^2}{\Delta x\Delta x} + \frac{z_{i(j+1)}^2 - 2z_{i(j+1)}z_{ij} + z_{ij}^2}{\Delta y\Delta y} - 4z_{ij}\right]\Delta x\,\Delta y
\end{aligned}$$

Now, if we have $\Delta x = \Delta y = 1/6$ (as in our case) then this formula will simplify to the following:

$$S = \sum_{j=0}^{5}\sum_{i=0}^{5}\left[z_{(i+1)j}^2 - 2z_{(i+1)j}z_{ij} + z_{i(j+1)}^2 - 2z_{i(j+1)}z_{ij} + 2z_{ij}^2 - (z_{ij}/9)\right]$$

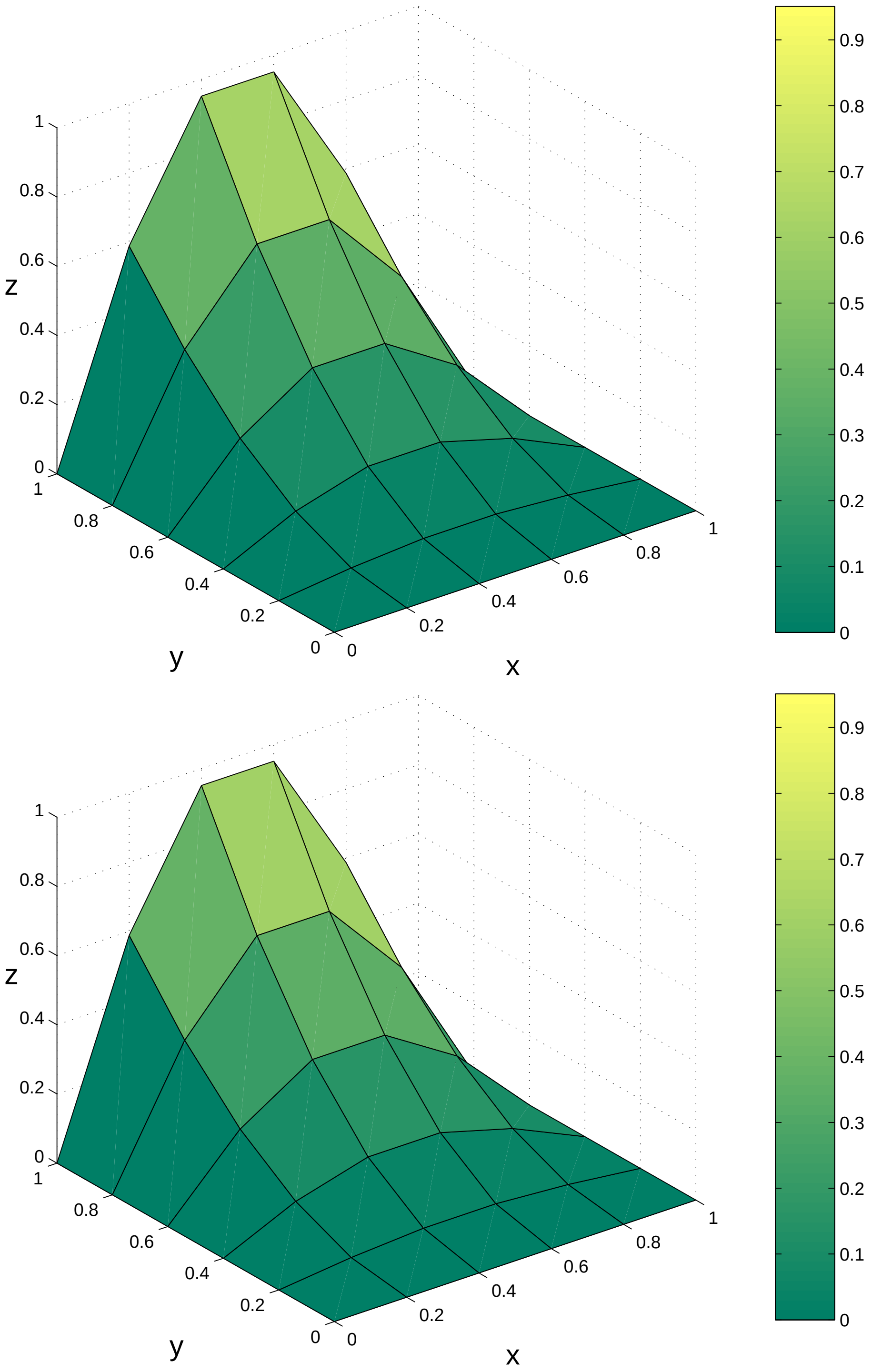

Figure 82: The upper frame is a plot of the points $z_{11}, \cdots, z_{44}$ which we obtained in Problem 7 of § 9 by finite difference (as well as the boundary points) while the lower frame is a plot of the analytical solution $z = y^2 \sin(\pi x)$ which we obtained in Problem 8 of § 1.6. For fair comparison, we use the same xy mesh in both plots.

that is:

$$\begin{aligned}
S \quad = \quad & z_{10}^2 - 2z_{10}z_{00} + z_{01}^2 - 2z_{01}z_{00} + 2z_{00}^2 - (z_{00}/9) + && (i=0, j=0) \\
& z_{20}^2 - 2z_{20}z_{10} + z_{11}^2 - 2z_{11}z_{10} + 2z_{10}^2 - (z_{10}/9) + && (i=1, j=0) \\
& z_{30}^2 - 2z_{30}z_{20} + z_{21}^2 - 2z_{21}z_{20} + 2z_{20}^2 - (z_{20}/9) + && (i=2, j=0) \\
& z_{40}^2 - 2z_{40}z_{30} + z_{31}^2 - 2z_{31}z_{30} + 2z_{30}^2 - (z_{30}/9) + && (i=3, j=0) \\
& z_{50}^2 - 2z_{50}z_{40} + z_{41}^2 - 2z_{41}z_{40} + 2z_{40}^2 - (z_{40}/9) + && (i=4, j=0) \\
& z_{60}^2 - 2z_{60}z_{50} + z_{51}^2 - 2z_{51}z_{50} + 2z_{50}^2 - (z_{50}/9) + && (i=5, j=0)
\end{aligned}$$

$$\begin{aligned}
& z_{11}^2 - 2z_{11}z_{01} + z_{02}^2 - 2z_{02}z_{01} + 2z_{01}^2 - (z_{01}/9) + && (i=0, j=1) \\
& z_{21}^2 - 2z_{21}z_{11} + z_{12}^2 - 2z_{12}z_{11} + 2z_{11}^2 - (z_{11}/9) + && (i=1, j=1) \\
& z_{31}^2 - 2z_{31}z_{21} + z_{22}^2 - 2z_{22}z_{21} + 2z_{21}^2 - (z_{21}/9) + && (i=2, j=1) \\
& z_{41}^2 - 2z_{41}z_{31} + z_{32}^2 - 2z_{32}z_{31} + 2z_{31}^2 - (z_{31}/9) + && (i=3, j=1) \\
& z_{51}^2 - 2z_{51}z_{41} + z_{42}^2 - 2z_{42}z_{41} + 2z_{41}^2 - (z_{41}/9) + && (i=4, j=1) \\
& z_{61}^2 - 2z_{61}z_{51} + z_{52}^2 - 2z_{52}z_{51} + 2z_{51}^2 - (z_{51}/9) + && (i=5, j=1)
\end{aligned}$$

$$\begin{aligned}
& z_{12}^2 - 2z_{12}z_{02} + z_{03}^2 - 2z_{03}z_{02} + 2z_{02}^2 - (z_{02}/9) + && (i=0, j=2) \\
& z_{22}^2 - 2z_{22}z_{12} + z_{13}^2 - 2z_{13}z_{12} + 2z_{12}^2 - (z_{12}/9) + && (i=1, j=2) \\
& z_{32}^2 - 2z_{32}z_{22} + z_{23}^2 - 2z_{23}z_{22} + 2z_{22}^2 - (z_{22}/9) + && (i=2, j=2) \\
& z_{42}^2 - 2z_{42}z_{32} + z_{33}^2 - 2z_{33}z_{32} + 2z_{32}^2 - (z_{32}/9) + && (i=3, j=2) \\
& z_{52}^2 - 2z_{52}z_{42} + z_{43}^2 - 2z_{43}z_{42} + 2z_{42}^2 - (z_{42}/9) + && (i=4, j=2) \\
& z_{62}^2 - 2z_{62}z_{52} + z_{53}^2 - 2z_{53}z_{52} + 2z_{52}^2 - (z_{52}/9) + && (i=5, j=2)
\end{aligned}$$

$$\begin{aligned}
& z_{13}^2 - 2z_{13}z_{03} + z_{04}^2 - 2z_{04}z_{03} + 2z_{03}^2 - (z_{03}/9) + && (i=0, j=3) \\
& z_{23}^2 - 2z_{23}z_{13} + z_{14}^2 - 2z_{14}z_{13} + 2z_{13}^2 - (z_{13}/9) + && (i=1, j=3) \\
& z_{33}^2 - 2z_{33}z_{23} + z_{24}^2 - 2z_{24}z_{23} + 2z_{23}^2 - (z_{23}/9) + && (i=2, j=3) \\
& z_{43}^2 - 2z_{43}z_{33} + z_{34}^2 - 2z_{34}z_{33} + 2z_{33}^2 - (z_{33}/9) + && (i=3, j=3) \\
& z_{53}^2 - 2z_{53}z_{43} + z_{44}^2 - 2z_{44}z_{43} + 2z_{43}^2 - (z_{43}/9) + && (i=4, j=3) \\
& z_{63}^2 - 2z_{63}z_{53} + z_{54}^2 - 2z_{54}z_{53} + 2z_{53}^2 - (z_{53}/9) + && (i=5, j=3)
\end{aligned}$$

$$\begin{aligned}
& z_{14}^2 - 2z_{14}z_{04} + z_{05}^2 - 2z_{05}z_{04} + 2z_{04}^2 - (z_{04}/9) + && (i=0, j=4) \\
& z_{24}^2 - 2z_{24}z_{14} + z_{15}^2 - 2z_{15}z_{14} + 2z_{14}^2 - (z_{14}/9) + && (i=1, j=4) \\
& z_{34}^2 - 2z_{34}z_{24} + z_{25}^2 - 2z_{25}z_{24} + 2z_{24}^2 - (z_{24}/9) + && (i=2, j=4) \\
& z_{44}^2 - 2z_{44}z_{34} + z_{35}^2 - 2z_{35}z_{34} + 2z_{34}^2 - (z_{34}/9) + && (i=3, j=4) \\
& z_{54}^2 - 2z_{54}z_{44} + z_{45}^2 - 2z_{45}z_{44} + 2z_{44}^2 - (z_{44}/9) + && (i=4, j=4) \\
& z_{64}^2 - 2z_{64}z_{54} + z_{55}^2 - 2z_{55}z_{54} + 2z_{54}^2 - (z_{54}/9) + && (i=5, j=4)
\end{aligned}$$

$$\begin{aligned}
& z_{15}^2 - 2z_{15}z_{05} + z_{06}^2 - 2z_{06}z_{05} + 2z_{05}^2 - (z_{05}/9) + && (i=0, j=5) \\
& z_{25}^2 - 2z_{25}z_{15} + z_{16}^2 - 2z_{16}z_{15} + 2z_{15}^2 - (z_{15}/9) + && (i=1, j=5) \\
& z_{35}^2 - 2z_{35}z_{25} + z_{26}^2 - 2z_{26}z_{25} + 2z_{25}^2 - (z_{25}/9) + && (i=2, j=5) \\
& z_{45}^2 - 2z_{45}z_{35} + z_{36}^2 - 2z_{36}z_{35} + 2z_{35}^2 - (z_{35}/9) + && (i=3, j=5) \\
& z_{55}^2 - 2z_{55}z_{45} + z_{46}^2 - 2z_{46}z_{45} + 2z_{45}^2 - (z_{45}/9) + && (i=4, j=5)
\end{aligned}$$

$$z_{65}^2 - 2z_{65}z_{55} + z_{56}^2 - 2z_{56}z_{55} + 2z_{55}^2 - (z_{55}/9) \qquad\qquad (i = 5, j = 5)$$

On eliminating the vanishing terms (from the zero boundary conditions) and simplifying we get:

$$\begin{aligned}
S \quad = \quad & 3z_{10}^2 + 4z_{11}^2 + 4z_{12}^2 + 4z_{13}^2 + 4z_{14}^2 + 4z_{15}^2 + z_{16}^2 + \\
& 3z_{20}^2 + 4z_{21}^2 + 4z_{22}^2 + 4z_{23}^2 + 4z_{24}^2 + 4z_{25}^2 + z_{26}^2 + \\
& 3z_{30}^2 + 4z_{31}^2 + 4z_{32}^2 + 4z_{33}^2 + 4z_{34}^2 + 4z_{35}^2 + z_{36}^2 + \\
& 3z_{40}^2 + 4z_{41}^2 + 4z_{42}^2 + 4z_{43}^2 + 4z_{44}^2 + 4z_{45}^2 + z_{46}^2 + \\
& 3z_{50}^2 + 4z_{51}^2 + 4z_{52}^2 + 4z_{53}^2 + 4z_{54}^2 + 4z_{55}^2 + z_{56}^2 + \\
& z_{60}^2 + z_{61}^2 + z_{62}^2 + z_{63}^2 + z_{64}^2 + z_{65}^2 \\
\\
& -2z_{20}z_{10} - 2z_{11}z_{10} - (z_{10}/9) \\
& -2z_{30}z_{20} - 2z_{21}z_{20} - (z_{20}/9) \\
& -2z_{40}z_{30} - 2z_{31}z_{30} - (z_{30}/9) \\
& -2z_{50}z_{40} - 2z_{41}z_{40} - (z_{40}/9) \\
& -2z_{60}z_{50} - 2z_{51}z_{50} - (z_{50}/9) \\
\\
& -2z_{21}z_{11} - 2z_{12}z_{11} - (z_{11}/9) \\
& -2z_{31}z_{21} - 2z_{22}z_{21} - (z_{21}/9) \\
& -2z_{41}z_{31} - 2z_{32}z_{31} - (z_{31}/9) \\
& -2z_{51}z_{41} - 2z_{42}z_{41} - (z_{41}/9) \\
& -2z_{61}z_{51} - 2z_{52}z_{51} - (z_{51}/9) \\
\\
& -2z_{22}z_{12} - 2z_{13}z_{12} - (z_{12}/9) \\
& -2z_{32}z_{22} - 2z_{23}z_{22} - (z_{22}/9) \\
& -2z_{42}z_{32} - 2z_{33}z_{32} - (z_{32}/9) \\
& -2z_{52}z_{42} - 2z_{43}z_{42} - (z_{42}/9) \\
& -2z_{62}z_{52} - 2z_{53}z_{52} - (z_{52}/9) \\
\\
& -2z_{23}z_{13} - 2z_{14}z_{13} - (z_{13}/9) \\
& -2z_{33}z_{23} - 2z_{24}z_{23} - (z_{23}/9) \\
& -2z_{43}z_{33} - 2z_{34}z_{33} - (z_{33}/9) \\
& -2z_{53}z_{43} - 2z_{44}z_{43} - (z_{43}/9) \\
& -2z_{63}z_{53} - 2z_{54}z_{53} - (z_{53}/9) \\
\\
& -2z_{24}z_{14} - 2z_{15}z_{14} - (z_{14}/9) \\
& -2z_{34}z_{24} - 2z_{25}z_{24} - (z_{24}/9) \\
& -2z_{44}z_{34} - 2z_{35}z_{34} - (z_{34}/9) \\
& -2z_{54}z_{44} - 2z_{45}z_{44} - (z_{44}/9) \\
& -2z_{64}z_{54} - 2z_{55}z_{54} - (z_{54}/9) \\
\\
& -2z_{25}z_{15} - 2z_{16}z_{15} - (z_{15}/9) \\
& -2z_{35}z_{25} - 2z_{26}z_{25} - (z_{25}/9)
\end{aligned}$$

$$\begin{aligned}
&-2z_{45}z_{35} - 2z_{36}z_{35} - (z_{35}/9)\\
&-2z_{55}z_{45} - 2z_{46}z_{45} - (z_{45}/9)\\
&-2z_{65}z_{55} - 2z_{56}z_{55} - (z_{55}/9)
\end{aligned}$$

On taking the partial derivatives of S with respect to $z_{11}, \cdots, z_{55}$ and setting the results to zero we get:

$$\begin{aligned}
\frac{\partial S}{\partial z_{11}} &= 8z_{11} - 2z_{10} - 2z_{21} - 2z_{12} - (1/9) = 0\\
\frac{\partial S}{\partial z_{12}} &= 8z_{12} - 2z_{11} - 2z_{22} - 2z_{13} - (1/9) = 0\\
\frac{\partial S}{\partial z_{13}} &= 8z_{13} - 2z_{12} - 2z_{23} - 2z_{14} - (1/9) = 0\\
\frac{\partial S}{\partial z_{14}} &= 8z_{14} - 2z_{13} - 2z_{24} - 2z_{15} - (1/9) = 0\\
\frac{\partial S}{\partial z_{15}} &= 8z_{15} - 2z_{14} - 2z_{25} - 2z_{16} - (1/9) = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial S}{\partial z_{21}} &= 8z_{21} - 2z_{20} - 2z_{11} - 2z_{31} - 2z_{22} - (1/9) = 0\\
\frac{\partial S}{\partial z_{22}} &= 8z_{22} - 2z_{21} - 2z_{12} - 2z_{32} - 2z_{23} - (1/9) = 0\\
\frac{\partial S}{\partial z_{23}} &= 8z_{23} - 2z_{22} - 2z_{13} - 2z_{33} - 2z_{24} - (1/9) = 0\\
\frac{\partial S}{\partial z_{24}} &= 8z_{24} - 2z_{23} - 2z_{14} - 2z_{34} - 2z_{25} - (1/9) = 0\\
\frac{\partial S}{\partial z_{25}} &= 8z_{25} - 2z_{24} - 2z_{15} - 2z_{35} - 2z_{26} - (1/9) = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial S}{\partial z_{31}} &= 8z_{31} - 2z_{30} - 2z_{21} - 2z_{41} - 2z_{32} - (1/9) = 0\\
\frac{\partial S}{\partial z_{32}} &= 8z_{32} - 2z_{31} - 2z_{22} - 2z_{42} - 2z_{33} - (1/9) = 0\\
\frac{\partial S}{\partial z_{33}} &= 8z_{33} - 2z_{32} - 2z_{23} - 2z_{43} - 2z_{34} - (1/9) = 0\\
\frac{\partial S}{\partial z_{34}} &= 8z_{34} - 2z_{33} - 2z_{24} - 2z_{44} - 2z_{35} - (1/9) = 0\\
\frac{\partial S}{\partial z_{35}} &= 8z_{35} - 2z_{34} - 2z_{25} - 2z_{45} - 2z_{36} - (1/9) = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial S}{\partial z_{41}} &= 8z_{41} - 2z_{40} - 2z_{31} - 2z_{51} - 2z_{42} - (1/9) = 0\\
\frac{\partial S}{\partial z_{42}} &= 8z_{42} - 2z_{41} - 2z_{32} - 2z_{52} - 2z_{43} - (1/9) = 0\\
\frac{\partial S}{\partial z_{43}} &= 8z_{43} - 2z_{42} - 2z_{33} - 2z_{53} - 2z_{44} - (1/9) = 0\\
\frac{\partial S}{\partial z_{44}} &= 8z_{44} - 2z_{43} - 2z_{34} - 2z_{54} - 2z_{45} - (1/9) = 0\\
\frac{\partial S}{\partial z_{45}} &= 8z_{45} - 2z_{44} - 2z_{35} - 2z_{55} - 2z_{46} - (1/9) = 0
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial S}{\partial z_{51}} &= 8z_{51} - 2z_{50} - 2z_{41} - 2z_{61} - 2z_{52} - (1/9) = 0 \\
\frac{\partial S}{\partial z_{52}} &= 8z_{52} - 2z_{51} - 2z_{42} - 2z_{62} - 2z_{53} - (1/9) = 0 \\
\frac{\partial S}{\partial z_{53}} &= 8z_{53} - 2z_{52} - 2z_{43} - 2z_{63} - 2z_{54} - (1/9) = 0 \\
\frac{\partial S}{\partial z_{54}} &= 8z_{54} - 2z_{53} - 2z_{44} - 2z_{64} - 2z_{55} - (1/9) = 0 \\
\frac{\partial S}{\partial z_{55}} &= 8z_{55} - 2z_{54} - 2z_{45} - 2z_{65} - 2z_{56} - (1/9) = 0
\end{aligned}
$$

On solving this system of simultaneous equations (with inserting the numeric values from the boundary points, i.e. $z_{10} = -1/36$, $z_{20} = -4/36$, etc.) we get:

$z_{11} = 0$	$z_{21} = -\frac{1}{18}$	$z_{31} = -\frac{1}{6}$	$z_{41} = -\frac{1}{3}$	$z_{51} = -\frac{5}{9}$
$z_{12} = \frac{1}{36}$	$z_{22} = 0$	$z_{32} = -\frac{1}{12}$	$z_{42} = -\frac{2}{9}$	$z_{52} = -\frac{5}{12}$
$z_{13} = \frac{1}{18}$	$z_{23} = \frac{1}{18}$	$z_{33} = 0$	$z_{43} = -\frac{1}{9}$	$z_{53} = -\frac{5}{18}$
$z_{14} = \frac{1}{12}$	$z_{24} = \frac{1}{9}$	$z_{34} = \frac{1}{12}$	$z_{44} = 0$	$z_{54} = -\frac{5}{36}$
$z_{15} = \frac{1}{9}$	$z_{25} = \frac{1}{6}$	$z_{35} = \frac{1}{6}$	$z_{45} = \frac{1}{9}$	$z_{55} = 0$

In fact, these values are identical to the values of the analytical solution (i.e. $z = xy - x^2$) at these points and hence we do not need to compare by plotting.

9. Make a brief comparison between the Rayleigh-Ritz method (of chapter 8) and the finite difference method (of chapter 9).

   Answer: We may note the following:

   • The cost of the finite difference method scales up as the mesh is refined (i.e. its size increases) while the cost of the Rayleigh-Ritz method is independent of the mesh size since it is based on obtaining a closed form. However, the Rayleigh-Ritz method also "scales up" as higher approximations are pursued (although this scaling up is not related to the mesh size).

   • The finite difference method is essentially based on algebra while the Rayleigh-Ritz method is essentially based on calculus (integration). The advantage of algebra over calculus is its simplicity (and hence the general availability of solution) while its disadvantage is that it is normally very lengthy and messy.

   • Both methods have flexibility and inflexibility with certain types of boundary conditions. So, each method has its advantages and disadvantages in this regard.

Chapter 10
Hybrid Methods

Hybrid methods are variational (and optimization) methods that are based on combining and mixing other (simpler) methods. In fact, the title of this chapter is very generic and hence it can include many methods which combine some of the previously-investigated methods (and possibly other methods). For example, we can combine the Rayleigh-Ritz method with a deterministic or stochastic computational technique to obtain a Rayleigh-Ritz numerical method (which is a hybrid method since it can be classified under chapter 8 and under chapter 9). In this case, the objective of the computational (numeric) method is to optimize the parameters (i.e. the parameters c_i's in the case of 1D and the parameters c_{ij}'s in the case of 2D and similarly for higher dimensions) of the Rayleigh-Ritz method in one go using an optimization algorithm based for example on conjugate gradient or Nelder-Mead or quasi-Newton or simulated annealing methods.[142] Anyway, the hybrid methods are generally very useful, flexible and practical and hence they represent an effective and powerful tool in the investigations and applications of the mathematics of variation in general (and the calculus of variations in particular). Moreover, they are usually associated with computer codes and packages and hence they are generally easy to use and easy to adapt and transform. Also, their cost is usually negligible because once the computer code or package is created or acquired it can be used infinite times with minimum time and effort (where the computers usually do all the hard work in the blink of an eye).

Problems

1. It is suggested in the text that the Rayleigh-Ritz method can be used in conjunction with a computer algorithm to obtain the parameters numerically. Suggest an alternative (hybrid) method in which the Rayleigh-Ritz method is used.
   Answer: For example, the analytical approach (which we explained and demonstrated in chapter 8) can be automated (stage by stage) to obtain an automated analytical (or symbolic) Rayleigh-Ritz method (rather than numeric Rayleigh-Ritz method which we suggested above). However, being a hybrid method may be disputed unless the automation is based on a different approach for obtaining the parameters.
2. Re-solve Problem 5 of § 8 using the y_5 Rayleigh-Ritz approximation and employing a numerical optimizer to obtain the parameters $c_1, \cdots, c_5$ (instead of differentiating I and solving the resulting equations analytically).
   Answer: We solved this Problem following a similar method to that used in Problem 5 of § 8 but instead of differentiating I and solving the resulting equations analytically we passed the expression of I to a numerical optimizer (to optimize I for the parameters $c_1, \cdots, c_5$) and we obtained: $c_1 \simeq 0.1680197269$, $c_2 \simeq -0.0522179496$, $c_3 \simeq 0.0062854969$, $c_4 \simeq 0.0041982436$ and $c_5 \simeq 0.0003880377$. These values are very close to the values obtained in Problem 5 of § 8. In fact, these values produce identical results (up to the sixth decimal place of the values of y_5) and identical plot.
3. Re-solve Problem 9 of § 8 using the z_{33} Rayleigh-Ritz approximation and employing a numerical optimizer to obtain the parameters $c_{11}, \cdots, c_{33}$ (instead of differentiating I and solving the resulting equations analytically).
   Answer: We solved this Problem following a similar method to that used in Problem 9 of § 8 but instead of differentiating I and solving the resulting equations analytically we passed the expression of I to a numerical optimizer (to optimize I for the parameters $c_{11}, \cdots, c_{33}$) and we obtained: $c_{11} = 4.0$, $c_{12} = -5.25$, $c_{13} = 5.25$, $c_{21} = -5.25$, $c_{22} = 15.75$, $c_{23} = -15.75$, $c_{31} = 5.25$, $c_{32} = -15.75$ and $c_{33} = 15.75$. These values are identical to the values obtained in Problem 9 of § 8 (noting the

---

[142] In fact, I used some of these (or similar) methods in my past research in fluid mechanics. However, these methods are associated with computer codes and hence they are beyond the scope of this book.

correspondence between $a, b, c, f, g, h, i, j, k$ and $c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, c_{33}$).

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Note: as well as the above references, we also consulted during our work on the preparation of this book many other books, research and review papers and general articles about this subject.

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